Operations Research

Linear Programming:

The name linear programming consist of the 2 important terms namely, 1) linear
2) Programming

The term linear means to the relationship between 2 Variables, which is of the form (Y = a+bx)
where a & b are Constants and x & Y are Variables
The term programming means planning a way of action in a Systematic manner with a view of achieving some desired optimum results.
That is maximization of profit, minimization of Cost etc.

Definition :-

"The, linear programming is defined as a programing of interdependent activities in a linear structure".
– By Dant-zig

Applications of linear programming :-

There are three important types of problem's Concerning Various feat's where linear programming technique Can be applied advantageously. They are :
① Problems of allocation
② Problems of assignments
③ Problems of transportation

Types of linear programming problems :

The linear programing problems are classified into 2 types they are : ① General linear programming problems
(or)
① Primial linear programing problems
② Duality linear programing problems

Procedure for primal linear programming

Problem:

STEPS:

① Formulation of the given problem

② Solution of the formulated problem

① Formulation of the given problem / L.P.P:

STEPS:

① Objective function

② Constraint function

③ non-negative function

→ Objective function

The objective of a problem may be either to maximize (or) minimize some result. In case of profit (or) income (or) output, the objective must be maximization but incase of "loss" (or) cost (or) input the objective must be minimized.

The objective function can be represented in the form $[Z_{(P)} = P_1x_1 + P_2x_2 + \ldots\ldots P_nx_n]$

where $Z_{(P)}$ = Maximum amount of profit

$P_1, P_2 \ldots\ldots P_n$ = Profit per different variable to be produced

$x_1, x_2 \ldots\ldots x_n$ = the no. of different variables to be produced under a decision

[or]

$[Z_{(C)} = C_1x_1 + C_2x_2 + \ldots\ldots + C_nx_n]$

where $Z_{(C)}$ = Minimum amount of ~~cost~~ cost

C_1, C_2 ... cost per Unit of the Variable

$x_1, x_2 \ldots\ldots x_n$ - The different no. of different Variables

② <u>Constraint Functions</u> :-

The, objective function is followed by the constraint function

① $a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n \leq b_1$

$\rightarrow$ Process - I

② $a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n \geq b_2$

$\rightarrow$ process - II

③ $a_{31}x_1 + a_{32}x_2 + \ldots\ldots + a_{3n}x_n = b_3 \rightarrow$ process - III

③ <u>Non - negative</u> functions :-

This, function implies that the production (or) perfomance of the Variables will never be negative.

i.e $\rightarrow x_1 \geq 0,\ x_2 \geq 0 \ldots\ldots x_n \geq 0$

Mathematical formulation of primial L.P.P :-

$$\max z = P_1x_1 + P_2x_2 + \ldots\ldots + P_nx_n$$

(or)

$$\min z = C_1x_1 + C_2x_2 + \ldots\ldots + C_nx_n$$

Subject to

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n \geq b_2$$

$$a_{31}x_1 + a_{32}x_2 + \ldots\ldots + a_{3n}x_n = b_3$$

(or)

$$x_1 \geq 0,\ x_2 \geq 0 \ldots\ldots x_n \geq 0.$$

Q) A problem produces 2 types of products through 2 processes i.e Foundary Process & Machine shop. the no of man powers required for each food unit of P & q in each of the process & the no. of man powers that can be avoided at best in the 2 processes are given as follows:

	Foundry Process	Machine Process
Product - P	10 Units	5 Units
Product - Q	6 Units	4 Units
Availability at best	1000 Units	600 Units

Sol:- Net Profit expected from each unit of the product are (P) ⟶ Rs 50 ₹ (q) → Rs 40 ₹

Formulate the problem for solution to arrive at the optimal no. of the 2 products (P & q) to be producted:

A) Step-1 ⟶ Notations :-

Let Z = total of maximum profit

Let x_1 = the no. of product 'p' to be produced

Let x_2 = the no. of product 'q' to be produced

Step-2 ⟶ Decision table :-

Product	Decision Variables	Foundry Process	Machinery Process	Profit (Rs)
P	x_1	10 Units	5 Units	50/-
q	x_2	6 Units	4 Units	40/-
Available at best		1000 Units	600 Units	

Step 3 → Construction of different linear Functions

i) Objective Functions

Maximum $Z = 50x_1 + 40x_2$

ii) Constraint Functions

$10x_1 + 6x_2 \leq 1000$

$5x_1 + 4x_2 \leq 600$

iii) Non-Negative Functions

$x_1 \geq 0$

$x_2 \geq 0$

Step 4 → Mathematical formulation of L.P.P :-

Max $Z = 50x_1 + 40x_2$

Subject to

$10x_1 + 6x_2 \leq 1000$

$5x_1 + 4x_2 \leq 600$

and $x_1, x_2 \geq 0$

Q) A foundary firm Contemplates to procure Special feats in a Combination which would provide the required vitamins Contents & Minimize the Cost as well. From the following data formulate the linear programming problem.

Feed	Units of Vitamins A, B, C in each Unit: A	B	C	Feed Cost [Rs]
P	4	1	0	2
Q	6	1	2	5
R	1	7	1	6
S	2	5	3	8

Minimum vitamin Contents needs per feed mix in Units

A - 12
B - 14
C - 8

Sol: Step ① → notations :

Let, Z = total amount of minimum Cost

Let, x_1 = the decision Variable for the feed P

let, x_2 = the decision Variable for the feed Q

let, x_3 = the decision Variable for the feed R

let, x_4 = the decision Variable for the feed S

Step ② → Decision table :-

Feed	Decision tables / Variables	Units of vitamins A,B,C in each Unit			Feed Cost (Rs)
		A	B	C	
P	x_1	4	1	0	2
Q	x_2	6	1	2	5
R	x_3	1	7	1	6
S	x_4	2	5	3	8
Minimum needed		12	14	8	-

Step ③ :- Construct of different linear functions

i) Objective Functions

Minimum $Z = 2x_1 + 5x_2 + 6x_3 + 8x_4$

ii) Constraint Functions

$4x_1 + 6x_2 + 1x_3 + 2x_4 \geq 12$

$1x_1 + 1x_2 + 7x_3 + 5x_4 \geq 14$

$0x_1 + 2x_2 + 1x_3 + 3x_4 \geq 8$

iii) Non - negative function :

$x_1 \geq 0$

$x_2 \geq 0$

Step (4) Mathematical Formulation of L.P.P :

$$\text{Min}(z) = 2x_1 + 5x_2 + 6x_3 + 8x_4$$

Subject to

$$4x_1 + 6x_2 + 1x_3 + 2x_4 \geq 12$$

$$1x_1 + 1x_2 + 7x_3 + 5x_4 \geq 14$$

$$0x_1 + 2x_2 + 1x_3 + 3x_4 \geq 8$$

and $x_1, x_2, x_3, x_4 \geq 0$

Q) A Manufacturer produces '2' types of models M_1 & M_2 each M_1 model requires 4 hours of traning & 2 hours of polyching where as each M_2 Model requires 2 hours of traning & 5 hours of polysching. The manufacturer has '2' Grinders & 3 polysehses. Each grinder works for 40 hours a week. 2 each polyscher works for (60) hours a Week.

Profit on m_1 model is [3₹]

Profit on m_2 model is (4₹)

whatever is produced in a week is sold in the market. how should the manufacturer allocate his production Capacity to have the 2 types of models. So that he may make the maximum in a Week.

Sol :- Step 1 : Notations :

Let z = Maximum amount of profit

Let x_1 =

Step 2 : Decision Table

Models	Decision Variables	Grinding	Polyeching	Profit
M_1	x_1	4	2	3
M_2	x_2	2	5	4
Maximum available		2 (40)	3(60)	-
		(80)	(180)	

Step 3 : Construction of Various different linear functions

i) Objective Function

Maximum $Z = 3x_1 + 4x_2$

iii) Constraint Function

$4x_1 + 2x_2 \leq 80$

$2x_1 + 5x_2 \leq 180$

(iii) Non - Negative Functions

$x_1 \geq 0$

$x_2 \geq 0$

Step 4 :- Mathematical Formulation of L.P.P :-

Max $Z = 3x_1 + 4x_2$

Subject to $4x_1 + 2x_2 \leq 80$

$2x_1 + 5x_2 \leq 180$

and $x_1, x_2 \geq 0$

Solution of the formulated LPP

→ After formulated a linear programming problem, the next step is to determine the values of the approval decision variables : x_1, x_2 etc

→ There are 3 types of Solutions of the linear programming problem namely : ① Feasible Solution
② Non Feasible Solution
③ Optimal Solution

① **Feasible Solution :**

A Solution which Satisfies the non-negative Conditions of a general linear programming problem is Called Feasible Solution

② **Non-Feasible Solution :-**

A Solution which Satisfies (not Satisfies) the non-negative Conditions of a primal linear programming problem is Called Non-Feasible Solution.

③ **Optimal Solution :-**

A feasible Solution which optimizes (max (or) min) the objective function of [G.L.P.P] is Called an Optimal Solution of a [P.L.P.P]

[G → General , P → Primal]

→ **Methods of Solution of L.P.P :-**

There are 2 methods of Solving a L.P.P. they are ① Graphing method
② Simplex Method

① Graphing Method

Step → ①

① First Consider the in-equalities as equalities

Step → ②

② Draw the lines in '2' dimensional plan

Eg:- x-axis, y-axis

Corresponding to each equation & non-negative restrictions

Step → ③

③ Find Convex region for the Values of the Variables which is the region bounded by the lines

Step → ④

④ Find a point in the Convex region, which gives the Optimum Solution.

Ex:- ① Find the Solution of the following L.P.P using Graphing method, maximize $Z = 3x_1 + 5x_2$

Subject to

$$x_1 + 2x_2 \leq 200$$
$$x_1 + x_2 \leq 150$$
$$x_2 \leq 60$$
and
$$x_1, x_2 \geq 0$$

Sol: Given that maximum $Z = 3x_1 + 5x_2$

Subject to

$$x_1 + 2x_2 \leq 200 \rightarrow ①$$
$$x_1 + x_2 \leq 150 \rightarrow ②$$
$$x_2 \leq 60 \rightarrow 300$$
and $x_1, x_2 \geq 0$

Now eq ① $\rightarrow x_1 + 2x_2 = 200$

If $x_1 = 0$ then $x_2 = 100 \rightarrow \left[\frac{200}{2}\right]$

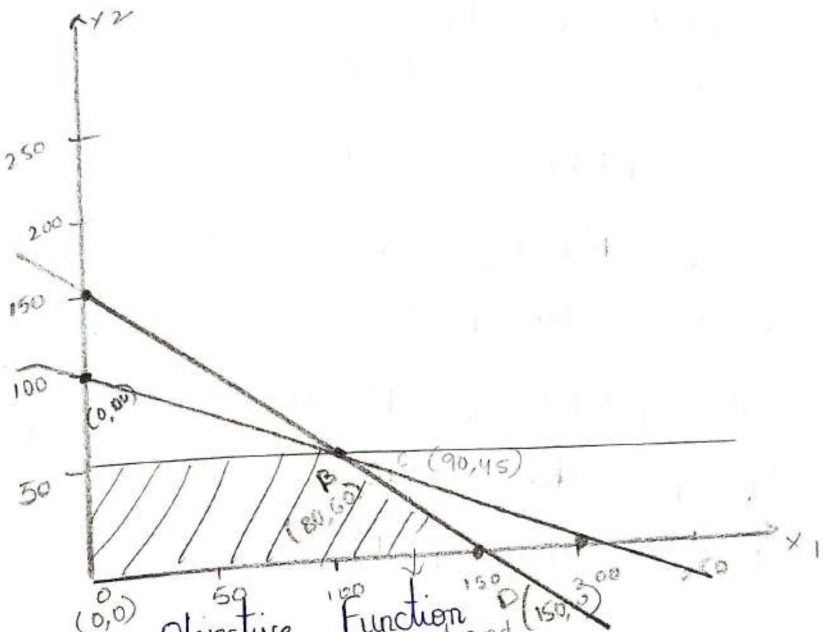
If $x_2 = 0$ then $x_1 = 200 \rightarrow \left[\frac{200}{1}\right]$

Now eq ② $\rightarrow x_1 + x_2 = 150$

If $x_1 = 0$ then $x_2 = 150 \left[\frac{150}{2}\right]$

If $x_2 = 0$ then $x_1 = 150 \Rightarrow \left[\frac{150}{1}\right]$

Now eq ③ $\Rightarrow x_2 = 60$



Points	Objective Function Maximum $Z = 3x_1 + 5x_2$
O - (0,0)	0
A - (0,60)	300
B - (80,60)	$\boxed{540}$ max
C - (90,45)	495
D - (150,0)	450

Note

① $3 \times 0 = 0$ + $5 \times 0 = 0$ } 0

② $3 \times 80 = 240$ + $5 \times 60 = 300$ } 540

2. Solve the following LPP by graphing method. Min Z $600x_1 + 400x_2$ Subject to

$3000x_1 + 1000x_2 \geq 24000$

$1000x_1 + 1000x_1 \geq 16000$

$2000x_1 + 6000x_2 \geq 48000$

and $x_1, x_2 \geq 0$

Given that

Min $Z = 600x_1 + 400x_2$

$\rightarrow 3000x_1 + 1000x_2 \geq 24000 \rightarrow (1)$

$1000x_1 + 1000x_1 \geq 16000 \rightarrow (2)$

$2000x_1 + 6000x_2 \geq 48000 \rightarrow (3)$

Now (1) $\Rightarrow 3000x_1 + 1000x_2 = 24000$

If $x_1 = 0$ Then $x_2 = 24$

If $x_2 = 0$ then $x_1 = 8$

Now (2) $\Rightarrow 1000x_1 + 1000x_2 = 16000$

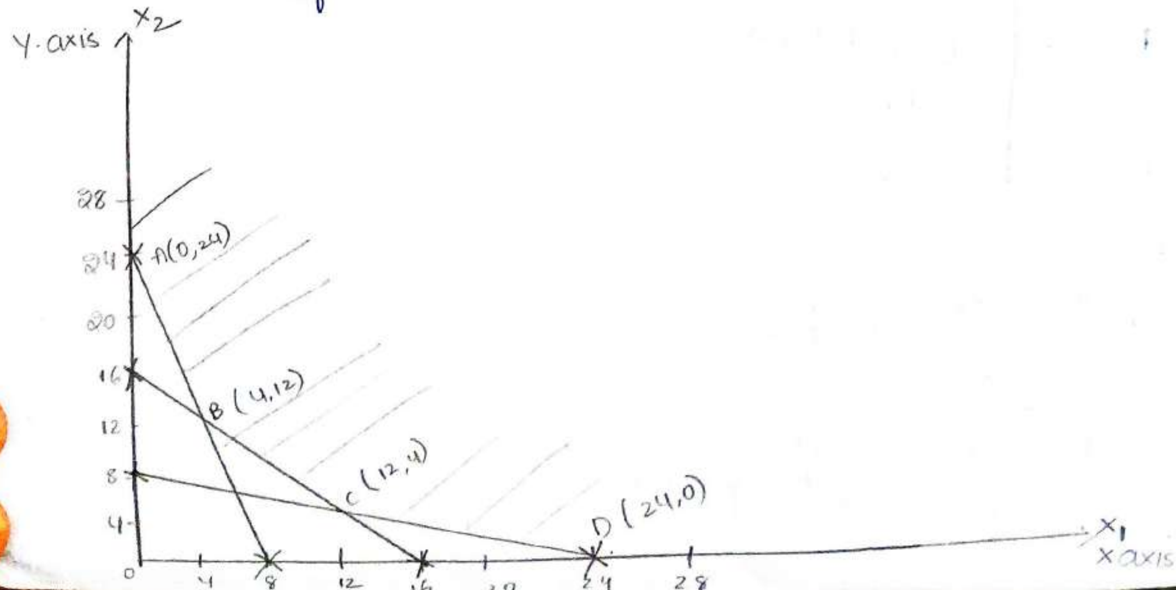
If $x_1 = 0$ then $x_2 = 16$

If $x_2 = 0$ then $x_1 = 16$

Now (3) $\Rightarrow 2000x_1 + 6000x_2 = 48000$

If $x_1 = 0$ then $x_2 = 8$

If $x_2 = 0$ then $x_1 = 24$



Corner Point	Objective Function Min $Z = 600x_1 + 400x_2$
A - (0, 24)	9600
B - (4, 12)	$\boxed{7200}$ min.
C - (12, 4)	8800
D - (24, 0)	14400

Q. Solve the following LPP through graphing method

Max $Z = 4X_1 + 6X_2$ Subject to $6X_1 + 2X_2 \le 48$

$X_1 + X_2 \le 16$

$X_1 + 3X_2 \le 24$

and $X_1, X_2 \ge 0$

Sol: Given that

Max $Z = 4X_1 + 6X_2$

$\longrightarrow$ $6X_1 + 2X_2 \le 48 \rightarrow (1)$

$X_1 + X_2 \le 16 \rightarrow (2)$

$X_1 + 3X_2 \le 24 \rightarrow (3)$

Now (1) $\longrightarrow 6X_1 + 2X_2 = 48$

If $X_1 = 0$ then $X_2 = 24$

If $X_2 = 0$ then $X_1 = 8$

(2) $\rightarrow X_1 + X_2 = 16$

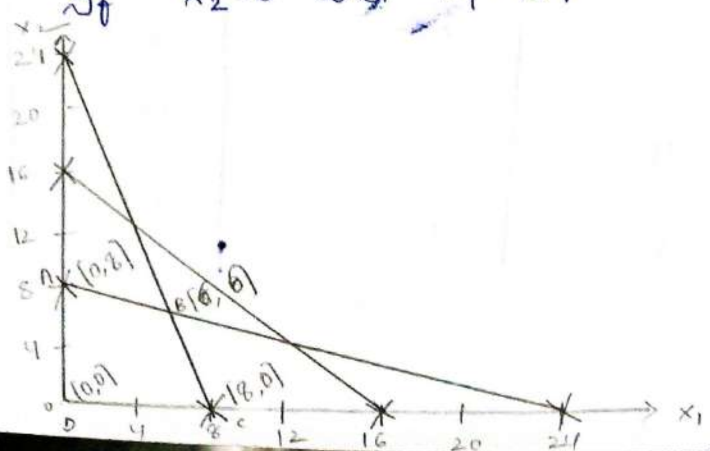
If $X_1 = 0$ then $X_2 = 16$

If $X_2 = 0$ then $X_1 = 16$

(3) $\longrightarrow X_1 + 3X_2 = 24$

If $X_1 = 0$ then $X_2 = 8$

If $X_2 = 0$ then $X_1 = 24$



Corner Points	Objective Functions: Max $Z = 4x_1 + 6x_2$
O - (0, 0)	0
A - (0, 8)	48
B - (6, 6)	$\boxed{60}$ max
C - (8, 0)	32

Q) A businessman manufactures chairs and tables. He has 1500 feet and 1050 feet of 2 types of Teak Wood and 810 hrs manpowers. For Manufacturing one Chair and one table, two feet and four feet of first type Teak-I, 5 feet and 8 feet of Teak-II type and two hrs and 4 hrs of man power are required. Expected profit on Table is ₹10 and chair is ₹5. To get maximum profit, how many tables and chairs that business man would need to manufactur. Solve through graphing method

Sol:- I Formulation of L.P.P

Steps ① Notations :-

Let Z = maximum amount of profit

x_1, x_2 = decision Variables

② Decision Variables

Product	Decision Variable	Teak type - I	Teak type - II	Man Power (hrs)	Profit
Chair	x_1	2	5	2	5
Table	x_2	4	8	4	10
Max. Available		1500	1050	810	

③ Construction of different linear functions

i) Objective function

$$\text{Max } Z = 5x_1 + 10x_2$$

ii) Constraint functions

$$2x_1 + 4x_2 \le 1500$$
$$5x_1 + 8x_2 \le 1050$$
$$2x_1 + 4x_2 \le 810$$

and ~~x~~

iii) Non Negative functions

$$x_1 \ge 0$$
$$x_2 \ge 0$$

④ iv) Mathematical formulation of L.P.P

$$\text{Max } Z = 5x_1 + 10x_2$$

Subject to

$$2x_1 + 4x_2 \le 1500$$
$$5x_1 + 8x_2 \le 1050$$
$$2x_1 + 4x_2 \le 810$$

and $x_1, x_2 \ge 0$

II Solution of the formulated L.P.P

Graphing method

Given that

$$\text{Max } Z = 5x_1 + 10x_2$$

Subject to: $2x_1 + 4x_2 \le 1500 \rightarrow ①$

$5x_1 + 8x_2 \le 1050 \rightarrow ②$

$2x_1 + 4x_2 \le 810 \rightarrow ③$

Now $2x_1 + 4x_2 = 1500$

If $x_1 = 0$ then $x_2 = 375$

If $x_2 = 0$ then $x_1 = 750$

② → $5x_1 + 8x_2 = 1050$

If $x_1 = 0$ then $x_2 = 131.25$

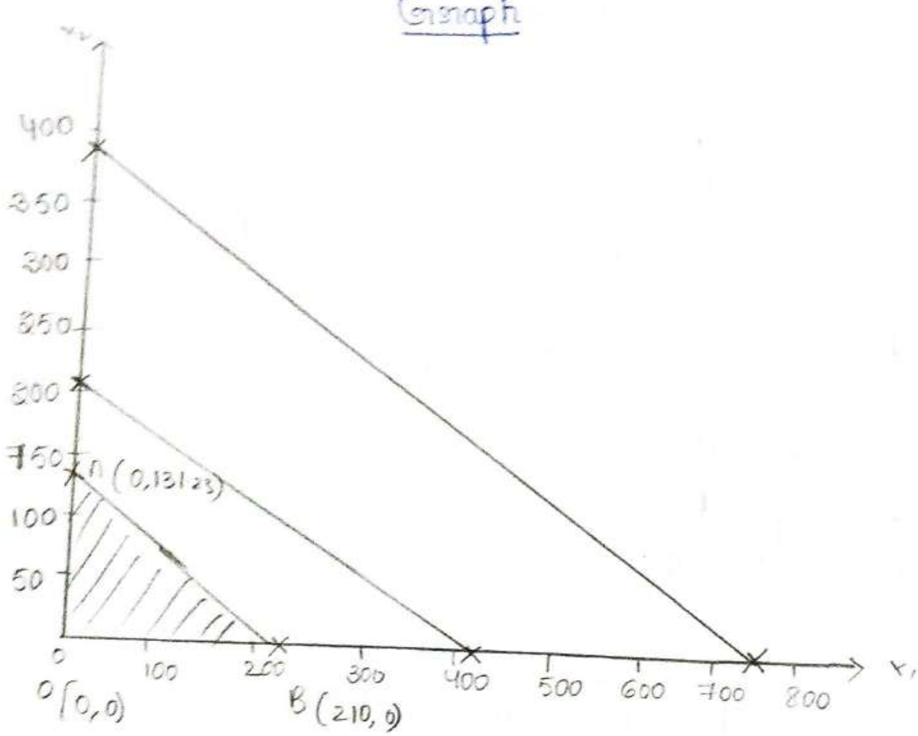
If $x_2 = 0$ then $x_1 = 210$

③ → $2x_1 + 4x_2 = 810$

If $x_1 = 0$ then $x_2 = 202.5$

If $x_2 = 0$ then $x_1 = 405$

Graph



Corner Points	Objective Functions Max $Z = 5x_1 + 10$
O (0, 0)	0
A (0, 131.25)	$\boxed{1312.5}$ max
B (210, 0)	1050

Q) Solve the following L.P.P by graphing method.

$$\text{Min } Z = 5x_1 + 8x_2$$

Subject to

$$x_1 + x_2 = 5$$

$$x_1 \leq 4$$

$$x_2 \geq 2 \quad \text{and} \quad x_1, x_2 \geq 0$$

Sol: Given that Min $Z = 5x_1 + 8x_2$

$x_1 + x_2 = 5 \longrightarrow (1)$

$x_1 < 4 \longrightarrow (2)$

$x_1 \geq 2 \longrightarrow (3)$

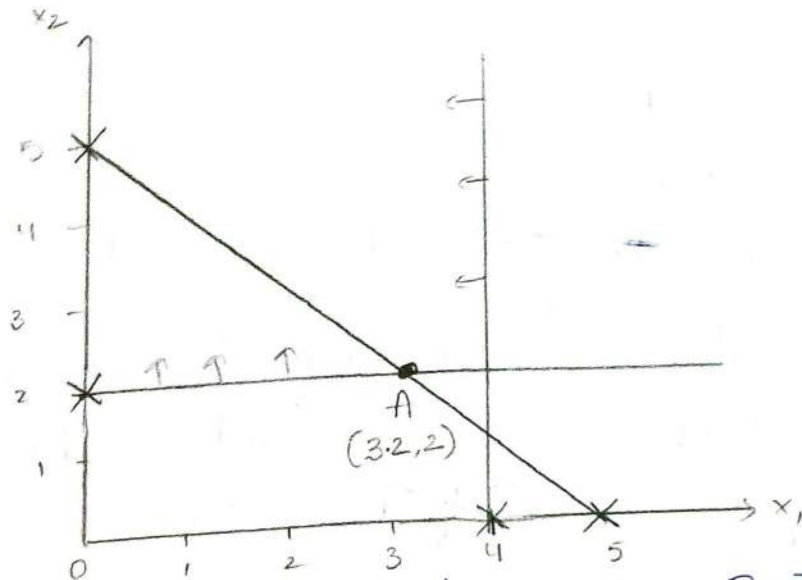
Now eq (1) $\Longrightarrow x_1 + x_2 = 5$

If $x_1 = 0$ then $x_2 = 5$

If $x_2 = 0$ then $x_1 = 5$

eq (2) $\Longrightarrow x_1 = 4$

eq (3) $\Longrightarrow x_2 = 2$



Corner Points	Objective Functions Min $Z = 5X_1 + 8X_2$
A (3.2, 2)	32 Min

→ **Slack Variables :**

The Variables that are used to Convert inequalities of the line less $\leq$ into equalities, it are Slack Variables

Eg.: If $3x_1 + 5x_2 \leq 14$

then

$3x_1 + 5x_2 + S_1 = 14$, where S_1 = Slack Variable

→ **Surplus Variables :**

The Variables that are used to Convert inequalities of the line $\geq$ into equalities are Surplus Variables

Eg: If $2x_1 + 4x_2 + 6x_3 \geq 18$

then

$2x_1 + 4x_2 + 6x_3 - S_1 = 18$ where S_1 = Surplus Variable

Constraints type	Variables to be introduced	The Coefficient of the Variable in objective function
$\leq$	S_1 S_2 ..... etc	If Max., $0 \cdot S_1$ If Min., $0 \cdot S_1$ ..... etc
$\geq$	$-S_1 + A_1$ $-S_2 + A_2$. etc,	If Max., $0 \cdot S_1 - M \cdot A_1$ If Min., $0 \cdot S_1 + MA_1$ etc
$=$	A_1 A_2 ... etc	If Max., $-M \cdot A_1$ If Min, $+M \cdot A_1$ etc

Q) Convert the following functions into the equations with the relevant Variables.

Max. $Z = x_1 + x_2$

Subject to $x_1 + 2x_2 \leq 4$
$2x_1 + 4x_2 \leq 6$ and $x_1, x_2 \geq 0$

Sol: $x_1 + 2x_2 + S_1 = 4$

$2x_1 + 4x_2 + S_2 = 6$

$\therefore$ Max. $Z = x_1 + x_2 + 0 \cdot S_1 + 0 \cdot S_2$

2. Convert the following functions into the equations with the relevant Variables

Min $Z = 3x_1 + 5x_2$

Subject to

$x_1 + x_2 \leq 4$

$2x_1 + 3x_2 \leq 8$

and $x_1, x_2 \geq 0$

Sol :-

$x_1 + x_2 + S_1 = 4$

$2x_1 + 3x_2 + S_2 = 8$

Min $Z = 3x_1 + 5x_2 + 0 \cdot S_1 + 0 \cdot S_2$

3. Convert the following functions into the equations with the relevant Variables.

Max $Z = 3x_1 + 5x_2$

S.T

$x_1 + x_2 \geq 4$

$2x_1 + 3x_2 \geq 8$

and $x_1, x_2 \geq 0$

Sol :-

$x_1 + x_2 - S_1 + A_1 = 4$

$2x_1 + 3x_2 - S_2 + A_2 = 8$

Max $Z = 3x_1 + 5x_2 + 0 \cdot S_1 - M \cdot A_1 + 0 \cdot S_E - M \cdot A_2$

4. Convert the following functions into the equations with the relevant Variables

Min $Z = 2x_1 + 3x_2$

S.T $x_1 + 4x_2 \geq 5$

$4x_1 + 6x_2 \geq 6$ and $x_1, x_2 \geq 0$

Sol :- $x_1 + 4x_2 - S_1 + A_1 = 5$

$4x_1 + 6x_2 - S_2 + A_2 = 6$

Min $Z = 2x_1 + 3x_2 + 0 \cdot S_1 + M \cdot A_1 + 0 \cdot S_2 + M \cdot A_2$

Q) Convert the following functions into the equations with the relevant Variables

$$\text{Max } Z = 2x_1 + 3x_2$$

St $x_1 + 4x_2 = 5$

$4x_1 + 6x_2 = 6$ and $x_1, x_2 \geq 0$

Sol) $x_1 + 4x_2 + A_1 = 5$

$4x_1 + 6x_2 + A_2 = 6$

$\therefore$ Max $Z = 2x_1 + 3x_2 - M \cdot A_1 - M \cdot A_2$

Q) Convert the following functions into the equations with the relevant Variables

$$\text{Min } Z = 4x_1 + 5x_2$$

Subject to

$3x_1 + 5x_2 = 14$

$5x_1 + 7x_2 = 16$ and $x_1, x_2 \geq 0$

Sol :- $3x_1 + 5x_2 + A_1 = 14$

$5x_1 + 7x_2 + A_2 = 16$

$\therefore$ Min $Z = 4x_1 + 5x_2 + M \cdot A_1 + M \cdot A_2$

Q) Convert the following functions into the equations with the relevant Variables

Min $Z = 4x_1 + 5x_2$

S.T

$3x_1 + 5x_2 \leq 14$ $\quad$ $2x_1 + 4x_2 = 18$

$5x_1 + 7x_2 \geq 16$ $\quad$ ~~and~~ and $x_1, x_2 \geq 0$

Sol: $3x_1 + 5x_2 + S_1 = 14$

$5x_1 + 7x_2 - S_2 + A_1 = 16$

$2x_1 + 4x_2 + A_2 = 18$

$\therefore$ Min $z = 4x_1 + 5x_2 + 0\cdot S_1 + 0\cdot S_2 + M\cdot A_1 + M A_2$

Q) Convert the following functions into the equations with the relevant Variables.

$$\text{Max} \cdot z = 3x_1 + 4x_2$$

S.T

$$3x_1 + 5x_2 \geq 4$$
$$5x_1 + 6x_2 = 6$$
$$6x_1 + 7x_2 \leq 8$$
and $x_1, x_2 \geq 0$

Sol:-

$$3x_1 + 5x_2 - S_1 + A_1 = 4$$
$$5x_1 + 6x_2 + A_2 = 6$$
$$6x_1 + 7x_2 + S_2 = 8$$

$$\text{Max } z = 3x_1 + 4x_2 + 0\cdot S_1 - M\cdot A_1 - M\cdot A_2 + 0\cdot S_2$$

<u>Simplex Method</u>

→ Solve the following L.P.P through Simplex Method

$$\text{Max } z = 4x_1 + 5x_2$$

Subject to

$$2x_1 + 3x_2 \leq 12$$
$$2x_1 + x_2 \leq 8$$
and $x_1, x_2 \geq 0$

Sol:- $2x_1 + 3x_2 + S_1 = 12$

$2x_1 + x_2 + S_2 = 8$ If $x_1 = 0$ And $x_2 = 0$ then

$\therefore$ IBFS $\begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = \begin{pmatrix} 12 \\ 8 \end{pmatrix}$ Initial Basic feasible Solution

Hence Max $z = 4x_1 + 5x_2 + 0\cdot S_1 + 0\cdot S_2$

Simplex Table 1.

C_B	X_B	Sol	x_1	x_2	S_1	S_2	θ
	$C_j \rightarrow$		4	5	0	0	
0	S_1	12	2	$\boxed{3}$ P.E	1	0	$\frac{12}{3} = 4 \rightarrow$
0	S_2	8	2	1	0	1	$\frac{8}{1} = 8$
	Z_j	0	0	0	0	0	
	$Z_j - C_j$	-	-4	-5	0	0	

Max

Simplex Table 2.

C_B	X_B	Sol	X_1	X_2	S_1	S_2	θ
	$C_j \rightarrow$		4	5	0	0	
5	X_2	4	$\frac{2}{3}$	1	$\frac{1}{3}$	0	$\frac{4}{\frac{2}{3}} = 6$
0	S_2	4	$\boxed{\frac{4}{3}}$ P.E	0	$\frac{-1}{3}$	1	$\frac{4}{\frac{4}{3}} = 3 \longrightarrow$
	Z_j	20	$\frac{10}{3}$	5	$\frac{5}{3}$	0	
	$Z_j - C_j$	-	$\frac{-2}{3}$	0	$\frac{5}{3}$	0	

$(1S_2 - X_2) = 1S_2$

	8	2	1	0	1
(-)	4	$\frac{2}{3}$	1	$\frac{1}{3}$	0
	4	$\frac{4}{3}$	0	$\frac{-1}{3}$	1

Simplex Table 3

C_B	X_B	Sol	X_1	X_2	S_1	S_2	θ
	$C_j \rightarrow$		4	5	0	0	
5	X_2	2	0	1	$\frac{1}{2}$	$\frac{-1}{2}$	
4	X_1	3	1	0	$\frac{-1}{4}$	$\frac{3}{4}$	
	Z_j	$\boxed{22}$	4	5	$\frac{3}{2}$	$\frac{1}{2}$	
	$Z_j - C_j$	-	0	0	$\frac{3}{2}$	$\frac{1}{2}$	

$\therefore$ All $(Z_j - C_j) \geq 0$

Verification

$X_1 = 3 \geq 0$

$X_2 = 2 \geq 0$

$\left(S_2 - \frac{3}{4}\right) = 4$

$X_1 \left(\frac{2}{3}\right)$

	4	$\frac{2}{3}$	1	$\frac{1}{3}$	0
(-)	2	$\frac{2}{3}$	0	$\frac{-1}{6}$	$\frac{1}{2}$
	2	0	1	$\frac{1}{2}$	$\frac{-1}{2}$

Max $z = 4x_1 + 5x_2$

$= 4(3) + 5(2)$

$= 22$

And $2(3) + 3(2) \le 12$

$2(3) + 2 \le 8$

Therefore the problem has feasible Solution

→ Max $Z = 3x_1 + 9x_2$

s.t $x_1 + 4x_2 \le 8$

$x_1 + 2x_2 \le 4$

and. $x_1, x_2 \ge 0$

Sol:

$x_1 + 4x_2 + S_1 = 8$

$x_1 + + 2x_2 + S_2 = 4$

$\therefore$ IBFS $\begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$

Hence Max $Z = 3x_1 + 9x_2 + 0S_1 + 0 \cdot S_2$

Simplex Table 1

	$C_j \rightarrow$		3	9	0	0	
C_B	X_B	Sol	x_1	x_2	S_1	S_2	θ
0	S_1	8	1	$\boxed{4}$ P.E	1	0	$\frac{8}{4} = 2 \rightarrow$
0	S_2	4	1	$\boxed{2}$	0	1	$\frac{4}{2} = 2$
	Z_j	0	0	0	0	0	
	$Z_j - C_j$	0	-3	-9 ↑	0	0	

$4 \times \frac{2}{3}$

$1 \cdot \frac{1}{2}$ $\frac{1}{2}$

$4 \times -\frac{1}{2}$ ②

$\frac{2}{3} \times \frac{1}{2}$

$\frac{4}{3} \times \frac{1}{2}$

$4 \times \frac{1}{2} = \frac{4}{2}$ ②

$\frac{2}{3} - \frac{1}{2}$ -4 $\frac{2}{3}$

$\frac{1}{6}$

$\frac{4}{2}$

$2 - \frac{1}{2} = -2$

$4 \times -\frac{1}{2}$

$4 \times \frac{1}{2} = \frac{4}{2}$

$= \frac{2}{3}$

		C_j	3	9	0	0	
C_B	X_B	Sol	x_1	x_2	S_1	S_2	θ
9	x_2	2	$\frac{1}{4}$	1	$\frac{1}{4}$	0	$\frac{2}{\frac{1}{4}} = 8$
0	S_2	0	$\boxed{\frac{1}{2}}$	0	$\frac{-1}{2}$	1	$\frac{0}{\frac{1}{2}} = 0 \rightarrow$
	Z_j	$\boxed{18}$	$\frac{9}{4}$	9	$\frac{9}{4}$	0	
	$Z_j - C_j$	-	$-\frac{3}{4}$	0	$\frac{9}{4}$	0	

$\uparrow$

4 1 2 0 1

multiply x_2 with (2) $\leftarrow$ 4 $\frac{1}{2}$ 2 $\frac{1}{2}$ 0

0 $\frac{1}{2}$ (0) $\frac{-1}{2}$ 1

		$C_j \rightarrow$	3	9	0	0	
C_B	X_B	Sol	X_1	X_2	S_1	S_2	θ
9	X_2	2	0	1	$\frac{1}{2}$	$\frac{-1}{2}$	
3	X_1	0	1	0	-1	2	
	Z_j	$\boxed{18}$	3	9	$\frac{3}{2}$	$\frac{3}{2}$	
		-	0	0	$\frac{3}{2}$	$\frac{3}{2}$	

2. $\frac{1}{4}$ 1 $\frac{1}{4}$ 0

Multiply S_2 with $\frac{1}{2}$ $\leftarrow$ 0 $\frac{1}{4}$ 0 $\frac{-1}{4}$ $\frac{1}{2}$

2 0 1 $\frac{1}{2}$ $\frac{-1}{2}$

$\therefore$ All $(Z_j - C_j) \geq 0$

Verification

$X_1 = 0 \geq 0$

$X_2 = 2 \geq 0$

and Max $Z = 3(0) + 9(2)$

$= 18$

Also $0 + 4(2) \leq 8$

$0 + 2(2) \leq 4$

(iii) If the given solution problem has feasible solution

Note: If the constraints of the form $\geq$ (or) $=$, then we use Big M method

→ The Big M method is also called penalty method

Problem: Use Big M method to maximize

$$\text{Max } Z = 2x_1 + x_2 + 3x_3$$

Subject to

$$x_1 + x_2 + 2x_3 \leq 5$$

$$2x_1 + 3x_2 + 4x_3 = 12$$

and $x_1, x_2 \geq 0$

Sol: Given that

$$x_1 + x_2 + 2x_3 + S_1 = 5$$

$$2x_1 + 3x_2 + 4x_3 + A_1 = 12$$

$\therefore$ Initial Basic feasible Solution : $\begin{pmatrix} S_1 \\ A_1 \end{pmatrix} = \begin{pmatrix} 5 \\ 12 \end{pmatrix}$

$\therefore$ $\text{Max } Z = 2x_1 + x_2 + 3x_3 + 0 \cdot S_1 - M \cdot A_1$

Simplex Table - 1

	$C_j \rightarrow$		2	1	3	0	$-M$	
C_B	X_B	Sol	x_1	x_2	x_3	S_1	A_1	θ
0	S_1	5	1	1	$\boxed{2}$ PE	1	0	$\frac{5}{2} = 2.5 \rightarrow$
$-M$	A_1	12	2	3	4	0	1	$\frac{12}{4} = 3$
	Z_j	$-12M$	$-2M$	$-3M$	$-4M$	0	$-M$	
	$C_j - Z_j$	-	$(-2M-2)$	$(-3M-1)$	$(-4M-3)$	0	0	

Simplex Table II

		c_j	2	1	3	0	-1	
C_B	x_B	Sol	x_1	x_2	x_3	S_1	A_1	θ
3	x_3	$\frac{5}{2}, \frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{\frac{5}{2}}{\frac{1}{2}} = 5$
$-M$	A_1	2	0	$\boxed{1}$	0	-2	1	$2 \rightarrow$
	z_j	$\frac{15}{2}-2M$	$\frac{3}{2}$	$\frac{3}{2}-M$	3	$\frac{3}{2}+2M$	$-M$	
	$z_j - c_j$	-	$-\frac{1}{2}$	$\frac{1}{2}-M$	0	$\frac{3}{2}+2M$	0	

↑

12 2 3 4 0 1
(-) 10 2 2 4 2 0
2 0 1 0 -2 1

Simplex Table - III

		c_j	2	1	3	0		
C_B	x_B	Sol	x_1	x_2	x_3	S_1	A_1	θ
3	x_3	$\frac{3}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$	$\frac{\frac{3}{2}}{\frac{1}{2}} = 3 \rightarrow$
1	x_2	2	0	1	0	-2	1	$\frac{2}{0} = \infty$
	z_j	$\frac{9}{2}+2$	$\frac{3}{2}$	1	$\frac{3}{2}$	$\frac{3}{2}-2$	4	
	$z_j - c_j$	-	$\frac{-1}{2}$	0	$\frac{-3}{2}$	$\frac{-1}{2}$	$4+M$	

↑

$\frac{5}{2}x + 3x$

$\frac{5}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ 1 $\frac{1}{2}$ 0

1 0 $\frac{1}{2}$ $\frac{1}{2}$ $-\frac{1}{2}$

$\frac{3}{2}$ $\frac{1}{2}$ 0 $-\frac{1}{2}$ $-\frac{1}{2}$ $-\frac{1}{2}$

$\frac{5}{2}+\frac{3}{2} = \frac{15}{2}-2M$

Simplex Table - IV

		$c_j \rightarrow$	2	1	3	0	$-M$	
C_B	x_B	Sol	x_1	x_2	x_3	S_1	A_1	θ
2	x_1	3	1	0	$\frac{1}{2}$	1	+2	
1	x_2	2	0	1	0	-2	-1	
	z_j	8	2	1	2	-0	-2	
	$z_j - c_j$	-	0	0	-1	-4	$-2+M$	

$\frac{1}{4} + 2 + 1 \times 0$ 2 2-2

$\therefore$ All $(z_j - c_j) \neq 0$

The Problem has non feasible Solution

Q) Use Big M method to minimize $Z = 4x_1 + 3x_2$

subject to

$2x_1 + x_2 \geq 10$

$-3x_1 + 2x_2 \leq 6$

$x_1 + x_2 \geq 6$

and $x_1, x_2 \geq 0$

Sol: $2x_1 + x_2 - S_1 + A_1 = 10$

$-3x_1 + 2x_2 + S_2 = 6$

$x_1 + x_2 - S_3 + A_2 = 6$

∴ Initial Basic feasible Solution : $\begin{pmatrix} A_1 = 10 \\ S_2 = 6 \\ A_2 = 6 \end{pmatrix}$

The given problem is in the form of minimization then first we convert them into maximization, then we can solve.

Now, $\text{Min } Z = -\text{Max}(-Z)$

$= -\text{Max } Z^*$

$\rightarrow \text{Max } Z^* = -\text{Min } Z$

$= -(4x_1 + 3x_2)$

$= -4x_1 - 3x_2$

Hence : $\text{Max } Z^* = -4x_1 - 3x_2 + 0 \cdot S_1 - M \cdot A_1 + 0 \cdot S_2 + 0 \cdot S_3 - M A_2$

Simplex Table - I

C_B	X_B	Sol	x_1	x_2	S_1	A_1	S_2	S_3	A_2	θ
	C_j		-4	-3	0	-M	0	0	-M	
-M	A_1	10	$\boxed{2}$ P.E	1	-1	1	0	0	0	$\frac{10}{2} = 5 \rightarrow$
0	S_2	6	-3	2	0	0	1	0	0	$\frac{6}{-3} = -2$
-M	A_2	6	1	1	0	0	0	-1	1	$\frac{6}{1} = 6$
	$Z_j \rightarrow$	$\boxed{-16M}$	-3M	-2M	M	-M	0	M	-M	
	$Z_j - C_j$	-	-3M+4 ↑	-2M+3	M	0	0	M	0	

Simplex Table II

C_B	x_B	Sol	x_1 (−4)	x_2 (−3)	S_1 (0)	A_1 (−M)	S_2 (0)	S_3 (0)	A_2 (−M)
−4	x_1	5	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0
0	S_2	21	0	$\frac{7}{2}$	$-\frac{3}{2}$	$\frac{3}{2}$	1	0	0
−M	A_2	1	0	PE $\boxed{\frac{1}{2}}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	−1	1
	z_j	−20−M	−4	$(-2-\frac{M}{2})$	$2-\frac{M}{2}$	$2+\frac{M}{2}$	0	M	−M
	z_j-c_j	−	0	$(1-\frac{M}{2})$ ↑	$(2-\frac{M}{2})$	$(2+\frac{3M}{2})$	0	M	0

6 −3 2 0 0 1 0 0

15 3 $\frac{3}{2}$ $\frac{-3}{2}$ $\frac{3}{2}$ 0 0 0

(+)

21 0 $\frac{7}{2}$ $\frac{-3}{2}$ $\frac{3}{2}$ 1 0 0

6 1 1 0 0 0 −1 1

5 1 $\frac{1}{2}$ $\frac{-1}{2}$ $\frac{1}{2}$ 0 0 0

1 0 $\frac{1}{2}$ $\frac{1}{2}$ $\frac{-1}{2}$ 0 −1 1

C_B	x_B	Sol	x_1 (−4)	x_2 (−3)	S_1 (0)	A_1 (−M)	S_2 (0)	S_3 (0)	A_2 (−M)	θ
−4	x_1	4	1	0	−1	1	0	1	−1	
0	S_2	14	0	0	−5	5	1	+7	−7	
−3	x_2	2	0	1	1	−1	0	$\frac{-1}{2}$	$\frac{1}{2}$	
	z_j	$\boxed{-22}$	−4	−3	7	−1	0	2	−2	
	z_j-c_j	−	0	0	7	−(1+M)	(0)	2	(−2+M)	

5 1 $\frac{1}{2}$ $\frac{-1}{2}$ $\frac{1}{2}$ 0 0 0

1 0 $\frac{1}{2}$ $\frac{1}{2}$ $\frac{-1}{2}$ 0 −1 1

4 1 0 $-\frac{1}{2}$ $\frac{7}{2}$ 1 $\frac{3}{2}+\frac{7}{2}$ 1 24

$\frac{10}{2}$

$-\frac{3}{2}-\frac{7}{2}$

$-\frac{10}{2}$ 5

21 0 $\frac{7}{2}$ $\frac{-3}{2}$ $\frac{3}{2}$ 1 0 0

7 0 $\frac{7}{2}$ $\frac{7}{2}$ $-\frac{7}{2}$ 0 −7 +7

(−) $2 \times \frac{7}{2}$

14 0 0 −5 5 1 +7 −7

$\therefore$ All $(z_j - c_j) \geq 0 \longrightarrow$ Verification

$x_1 = 4 \geq 0$

$x_2 = 2 \geq 0$

$\Rightarrow$ Now,

$2x_1 + x_2 \geq 10$	$-3x_1 + 2x_2 \leq 6$	$x_1 + x_2 \geq 6$
$2(4) + 2 \geq 10$	$-3\times4 + 2(2) \leq 6$	$4 + 2 \geq 6$
$8 + 2 \geq 10$	$-12 + 4 \leq 6$	$6 \geq 6$
$10 \geq 10$	$-8 \leq 6$	

Now $\quad \text{Max } z^* = -\text{Min } z$

$\text{Min } z = -\text{Max } z^*$

$= -(-22)$

$= 22$

Procedure :-

Simplex Method - Algorithms - Steps .

① Check whether the objective function of the given L.P.P is to be maximized (or) minimized. If it is to be minimized then they Convert it into a problem of maximizing it by using $\text{Min } z = -(\text{Max}(-z))$

② Check whether all b_i, i is taking as 1 to m (or) non negative. If any one of b_i is negative, then multiply the Corresponding equation of Constraints of -1. So, to get all $b_i \geq 0$

③ Convert all the inequations of the Constraints into equations by introducing Slack and / or Surplus Variables Constraints put the Cost of these Variables = 0

④ Obtain an intial basic feasible Solution [IBFS] to the problem in the form X_B and put it in the second column of the simplex table

5 → Calculate the net evaluation $(z_j - c_j)$, $j = 1$ to n. Examine the sign of $(z_j - c_j)$

(i) If all $(z_j - c_j) \geq 0$, then the IBFS $[x_B, \text{sol}]$ is optimum basic feasible solution

(ii) If atleast $(z_j - c_j) < 0$, we proceed the next step

6 → If there are more than one negative of $(z_j - c_j)$ then choose the most negative of them, let it be $(z_r - c_r)$, for $j = r$

i) If all $a_{ir} \leq 0$, there is an unbounded solution to the given problem

ii) If atleast 1 $a_{ir} > 0$, then the corresponding vector a_r enters to the basis x_B

7 → Calculate the ratios $\left[\frac{\text{sol}}{a_{ir}}\right]$ and choose the minimum of them, then the vector will leaves the basis x_B. The common element most negative of $z_j - c_j$ and minimum of θ is known as pivotal element

8 → Convert the pivotal element to 1 by dividing its row by the pivotal element itself and all other elements in its column to 0's.

9 → Go to step 5 and repeat the calculation procedure either all $(z_j - c_j) \geq 0$, an optimum solution is obtained or there is an indication of an unbounded solution

UNIT - II

Transportation Problems

The transportation Problem is one of the Sub classes of linear programming problems, in which the objective is to transport Various quantities of a single homogenous Commodity that are initially ~~sold~~ Stored at Various horizons [Go downs], to different destinations [Shops, in such a way] that the total transportation Cost is minimum.

→ Mathematical Formulation of the transportation problem:

A transportation problem Can be stated mathematically as a LPP as follows :-

$$\text{Min } z = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \cdot C_{ij}$$

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1 \text{ to } m$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1 \text{ to } n$$

and

$x_{ij} \geq 0$ for $\forall$ i & j where a_i = quantity of Commodity available at horizon height. b_j = quantity of Commodity leaded at destination j. C_{ij} = task of transporting one unit of Commodity from i^{th} horizon to j^{th} destination

x_{ij} = quantity transported from the horizon i to destination j.

Note :- The necessary and Sufficient Condition for the existence of a feasible Solution through a transportation

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

Note 2 : The total number of basic Variables in an $m \times n$ transportation table $(m+n-1)$.

Methods for finding ibfs of transportation Problem

To find initial basic feasible Solution [ibfs] for a transportation problem, We describe 3 most Commonly used methods.

i) North-West Corner Rule method (or) NWC method

ii) Least Cost method (or) Matrix Minima Method.

iii) Vogel's approximation method [VAM] (or) Penalty method

→ North West Corner Rule Method :

Steps :-

1. To start with the Cell at the Upper left in (North-West) Corner of the transportation matrix, we allocate as much as possible. So that either the Capacity of the first row is exhausted (or) the destination requirement of the first Column is Satisfied i.e $x_{11} = \min(a_1, b_1)$

2. If b_1 less than A_1

to the Second Column and may ~~and~~ Second allocation of magnitude $x_{12} = \min\{(a_1 - x_{11}), b_2\}$ in the Cell [1,2]

If $b_1 > A_1$, We move Down Vertically to the Second row and may the Second allocation of magnitude $x_{21} = \min.\{a_2, (b_1 - x_{11})\}$ in the Cell (1,2).

→ If $b_1 = A_1$ then there is a tie for the Second allocation. One Can may the Second allocation of magnitude $x_{12} = \text{Min}.\{(a_1 - a_1), b_1\} = 0$ in the Cell [1,2)

(or) $x_{21} = \text{Min}\{~~(b_1-b_1), a_1~~\}\{a_2, (b_1-b_1)\} = 0$ in the Cell [2,1]

Step 3 :- Repeat steps 1 and 2 moving down towards the lower right Corner of the transportation table, all the rim requirements are Satisfied.

Eg :- Obtain an Intial basic feasible Solution to the transportation problem. by using North West Corner rule. method.

	I	II	III	IV	Available
A	11	13	17	14	250
B	16	18	14	10	300
C	21	24	13	10	400
Requirement	200	225	275	250	

Sol :-

	I	II	III	IV	Available
A	200 ~~11~~	13	17	14	~~250~~ 50
B	~~16~~	18	14	10	300
C	~~21~~	24	13	10	400
Requirement	~~200~~	225	275	250	950 / 950

$\Sigma a_i = \Sigma b_j$)

	II	III	IV	Availability
A	$\boxed{50}$ ~~13~~	~~17~~	~~14~~	~~50~~
B	18	14	10	300
C	24	13	10	400
Req	~~225~~ 175	275	250	750 / 750

	II	III	IV	Avail
B	$\boxed{175}$ ~~18~~	14	10	~~300~~ 125
C	~~24~~	13	10	400
Requirements	~~175~~	275	250	700 / 700

	III	IV	Availabil
B	$\boxed{125}$ ~~14~~	~~10~~	~~125~~
C	13	10	400
Req	~~275~~ 150	250	525 / 525

	III	IV	Avail
C	$\boxed{150}$ 13	10	~~400~~ 250
Req	150	250	400 / 400

	IV	Avail
C	$\boxed{250}$ 10	250
Req		

	IV	Avail
C	$\boxed{250}$ 10	~~250~~
Requirem	~~250~~	250 / 250

Min Cost =
$(200 \times 11) +$
$(500 \times 13) +$
$(175 \times 18) +$
$(125 \times 14) +$
$(150 \times 13) + (250 \times 10)$ $= 12200$ $\square$

Q. Obtain a basic feasible solution of a transportation problem whose cost and requirements table is given below

Origin	D_1	D_2	D_3	Supply
O_1	2	7	4	5
O_2	3	3	1	8
O_3	5	4	7	7
O_4	1	6	2	14
Demand	7	9	18	

Sol :- Given that

Origin	D_1	D_2	D_3	Supply
O_1	2	7	4	5
O_2	3	3	1	8
O_3	5	4	7	7
O_4	1	6	2	14
Demand	7	9	18	$\frac{34}{34}$

Therefore : $\sum_{i}^{m} a_i = \sum_{j=1}^{n} b_j$, The given transportation problem is balanced.

Origin	D_1	D_2	D_3	Supply
~~O_1~~	$\boxed{5}$ ~~2~~	~~7~~	~~4~~	~~5~~
O_2	3	3	1	8
O_3	5	4	7	7
O_4	1	6	2	14
Demand	~~7~~ 2	9	18	$\frac{34}{34}$

Origin	Destination			Supply
	D_1	D_2	D_3	
O_2	2 | ~~3~~	3	1	~~8~~ 6
O_3	~~5~~	4	7	7
O_4	~~1~~	6	2	14
Demand	~~2~~	9	18	29 / 29

Origin	Destination		Supply
	D_2	D_3	
~~O_2~~	6 | 3	1	~~6~~
O_3	4	7	7
O_4	6	2	14
Demand	~~9~~ 3	18	27 / 27

Origin	Destination		Supply
	D_2	D_3	
O_3	3 | ~~4~~	7	~~7~~ (4)
O_4	~~6~~	2	14
Demand	~~3~~	18	21 / 21

Origin	Destination	Supply
	D_3	
~~O_3~~	4 | ~~7~~	~~4~~
O_4	2	14
Demand	(14) ~~18~~	18 / ~~18~~

Origin	Destination	Supply
O_4	$\boxed{14}$ 2	~~14~~
Demand	~~14~~	14 / 14

$\therefore$ Min. Cost $= (5\times2) + (2\times3) + (6\times3) + (3\times4) + (4\times7) + (14\times2)$

$= 102$ $\square$

Origin	Destination	Supply.
O_1	14 | 2	14
Demand	~~14~~	14 | 14

$\therefore$ Min. Cost $= (5\times2) + (2\times3) + (6\times3) + (3\times4) + (4\times1) + (14\times2)$

$= 102$ □

→ Use NWC method, Solve the following transportation problem.

	D_1	D_2	D_3	D_4	Available
O_1	6	4	1	5	14
O_2	8	9	2	7	16
O_3	4	3	6	2	5
Requirements	6	10	15	4	

Sol:

	~~D_1~~	D_2	D_3	D_4	Available
O_1	6 | ~~6~~	4	1	5	~~14~~ 8
O_2	~~8~~	9	2	7	16
O_3	~~4~~	3	6	2	5
Requirements	~~6~~	10	15	4	35 \ 35

→ (2)

	D_2	D_3	D_4	Available
~~O_1~~	~~8~~ | ~~4~~	~~1~~	~~5~~	~~8~~
O_2	9	2	7	16
O_3	3	6	2	5
Requirements	~~10~~ 2	15	4	29 \ 29

→ (3)

	~~D_2~~	D_3	D_4	Available
O_2	~~2|9~~	2	7	~~16~~ (14)
O_3	~~3~~	6	2	5
Requirements	~~2~~	15	4	21 \ 21

→ (4)

	D_3	D_4	A
~~O_2~~	~~14|2~~	~~7~~	~~14~~
O_3	6	2	5
R	~~15~~ 1	4	19 \ 19

(5) →

	~~D_3~~	D_4	A
O_3	~~1|6~~	2	~~5~~ 4
R	~~1~~	4	5 \ 5

(6) →

	D_4	A
O_3	4|2	4
R	4	$\boxed{4 \backslash 4}$

$\therefore$ Min Cost :- $6\times6 + 4\times8 + 2\times9 + 14\times2 + 1\times6 + 4\times2$

$= 128$ □

Note :- (1) The allocated Cells in the transportation table are Called Occupied Cells and empty Cells are Called Non Occupied Cells.

(2) If tie Occurs $[a_i = b_j]$, either row or Column may be exhausted but not both only one

→ Unbalanced transportation Problem

The total available Supply is not equal to the total requirements, ~~Subs~~ such type of transportation problems are Called Unbalanced transportation Problems

$$\Sigma a_i \neq \Sigma b_j$$

sol

Warehouse

* Plant	D	E	F	G	H	Capacity
A	5	8	6	6	3	800
B	4	7	7	6	5	500
C	8	4	6	6	4	900
Requirements	400	400	500	400	800	2200 / 2500

	D	E	F	G	H	Capacity
A	400 | ~~5~~	8	6	6	3	~~800~~ 400
B	~~4~~	7	7	6	5	500
C	~~8~~	4	6	6	4	900
C	~~0~~	0	0	0	0	300
Requirements	~~400~~	400	500	400	800	2500 / 2500

⇒ 3

	E	F	G	H	Capacity
A	~~8~~	~~6~~	~~6~~	~~3~~	~~400~~
B	7	7	6	5	500
C	4	6	6	4	900
C	0	0	0	0	300
Req	~~400~~ 0	500	400	800	2100 / 2100

	~~E~~	F	G	H	Capacity
B	0 ~~A~~	7	6	5	500
C	~~4~~	6	6	4	900
C	~~0~~	0	0	0	300
Requirement	~~0~~	500	400	800	1700 / 1700

	F	G	H	Capacity
~~B~~	500 ~~4~~	~~6~~	~~5~~	~~500~~
C	6	6	4	900
C	0	0	0	300
Requirement	~~500~~ 0	400	800	1700 / 1700

⇒

	~~F~~	G	H	Capacity
C	0 ~~6~~	6	4	900
C	~~0~~	0	0	300
Req	~~0~~	400	800	1200 / 1200

⇒

	G	H	Capacity
C	400 ~~6~~	4	~~900~~ 500
C	~~0~~	0	300
Req	~~400~~	800	1200 / 1200

⇒

	~~G~~	H	Capacity
~~C~~		~~4~~	~~500~~
C		0	300
		~~800~~ 300	800 / 800

" Capacity,

c 3rd 0 1300

Req 300 ' 300
300

$\rightarrow$ Min (z) = (400×5) + (400×8) + (0×1)×
(500×1)+ (0×6) + (400×6) +
(500×4) + (300×0)
= 13100 $\square$

$\rightarrow$ Solve the following transportation problem using North-West Corner Method

Q)

From	Dealer 1	2	3	4	Plant Capacity
Plant-1	7	10	14	8	30
Plant-2	7	12	12	6	40
Plant-3	5	8	15	9	50
Plant-4	20	20	25	30	95 | 120

Sol. Given that

$\therefore \Sigma a_i \neq \Sigma b_j$,

The given transportation problem is unbalanced. First we convert it into balanced, Then we can Solve

From	Dealer 1	2	3	4	5	Plant Capacity
Plant-1	$\boxed{20}$ 7	10	14	8	0	~~30~~ 10
Plant-2	7	12	12	6	0	40
Plant-3	5	8	15	9	0	50
Dealer Demand	~~20~~	20	25	30	25	120 | 120

⟹

From	Dealer 2	3	4	5	Plant Capacity
~~Plant 1~~	~~10] 10~~	~~14~~	~~8~~	~~0~~	~~10~~ 0
Plant 2	12	12	6	0	40
Plant 3	8	15	9	0	50
Dealer Demand	~~20~~ 10	25	30	25	100 / 100

⟹

From	Dealer ~~2~~	3	4	5	Plant Capacity
Plant - 2	10] ~~12~~	12	6	0	~~40~~ 30
Plant - 3	~~8~~	15	9	0	50
Dealer demand	~~10~~	25	30	25	90 / 90

⟹

From	Dealer ~~3~~	4	5	Plant Capacity
Plant - 2	25] ~~12~~	6	0	~~30~~ 5
Plant - 3	~~15~~	9	0	50
Dealer demand	~~25~~	30	25	80 / 80

⟹

From	4	5	Plant Capacity
~~Plant - 2~~	5] ~~6~~	~~0~~	~~5~~
Plant - 3	9	0	50
Dealer demand	~~30~~ 25	25	55 / 55

From	~~4~~	5	Plant Capacity
Plant 3	25 | ~~7~~	0	50 25
Dealer demand	~~25~~	25	$\cancel{50}$ 50

From	5	Plant Capacity
Plant 3	25 | 0	25
D.D	25	$\cancel{25}$ 25

$$\text{Min } z = 20\times7 + 10\times10 + 10\times12 + 25\times12 + 5\times6 + 25\times9.$$
$$= 915 \square$$

⟹ Least Cost Method (or) Matrix Minima Method

Steps :- 1. Identify the smallest Cost in the Cost matrix of the transportation table then it be C_{ij}. Allocate x_{ij} = minimum (a_i, b_j) in the Cell (i, j)

Step 2 :- If $x_{ij} = a_i$, Cross of the i^{th} row of the transportation table and degrees b_j by a_i. Go to

Step 3 :- If $x_{ij} = b_j$, Cross of the Column j^{th} of the transportation table and degrees a_i by b_j. Go to step 3.

If $x_{ij} = a_i = b_j$, Cross of the either i^{th} row of j^{th} Column. But not both

Step 3 :- Repeat steps 1 and 2 for resulting reducing transportation table Untill all the rim requirements are Satisfied. Whenever the minimum Cost is not Unique make an ~~Orbit~~ Arbitary Choice among the minimum

Eg:- Obtain an Initial basic feasible Solution to the following transportation problem using the least Cost method

	D_1	D_2	D_3	D_4	Supply
O_1	1	2	3	4	6
O_2	4	3	2	0	8
O_3	0	2	2	1	10
Demand	4	6	8	6	24 / 24

Sol:- Given that

$\therefore \Sigma a_i = \Sigma b_j$ the given transportation Problem is balanced

$\Rightarrow$

	D_1	D_2	D_3	$\cancel{D_4}$	Supply
O_1	1	2	3	4	6
O_2	4	3	2	$\underline{6}$ 0	$\cancel{8}$. 2
O_3	0	2	2	$\cancel{1}$	10
D	4	6	8	$\cancel{6}$	

$\Rightarrow$

	$\cancel{D_1}$	D_2	D_3	Supp
O_1	1	2	3	6
O_2	4	3	2	2
O_3	$\underline{4}$ $\cancel{0}$	2	2	$\cancel{10}$ 6
D	$\cancel{4}$	6	8	

	D_2	D_3	Supply
O_1	$6\rfloor \cancel{2}$	3	$\cancel{6}$ 0
O_2	$\cancel{3}$	2	2
O_3	$\cancel{2}$	2	6
D	$\cancel{6}$	8	14 \ 14

	D_3	Supply
O_1	3	0
O_2	2	2
$\cancel{O_3}$	$6\rfloor \cancel{2}$	$\cancel{6}$
D	$\cancel{8}$ 2	8 \ 8

	D_3	Supply
O_1	$\cancel{3}$	0
O_2	$2\rfloor \cancel{2}$	$\cancel{2}$ 0
D	$\cancel{2}$	2 \ 2

$\cancel{O_1}$	0
$\cancel{O_2}$	0
$\cancel{D}$	0

	D_3	Supply
O_1	3	0
$\cancel{O_2}$	$2\rfloor \cancel{2}$	$\cancel{2}$
	$\cancel{2}$ 0	2 | 2

	D_3	Supp
O_1	$0\rfloor 3$	0
Re	0	0 \ 0

$$\Rightarrow \text{Min } Z = 6\times 0 + 0\times 4 + 2\times 6 + 2\times 6 + 2\times 2 + 3\times 0$$

$$= 28 \ \square$$

→ Use Matrix Minima Method, Find an Initial basic feasible solution to the transportation problem as given below

→ Vogel's Approximation Method (or) Penalty Methods

① → Calculate Penalties by taking differences between the Minimum & Next to Transportation Costs in each row & each Column

② → Allocate as much as possible in the lowest Cost cell of row (or Column) having a Circled row [or Column] difference.

③ → Circle the largest row difference or Column difference. In the event of choose either.

④ → Incase the allocation is made fully to a row [or Column], ignore that row [or Column] for further Consideration By Cross it.

⑤ → Revise the differences again and Cross Out the earlier figures. Go to step 2 and Step 3.

⑥ → Continue the procedure until all rows and Columns have been Crossed out. i.e distribution is Complete.

Q) Problem :- Use Vogel's Approximation method to obtain an intial basic feasible Solution of transportation problem.

	D	E	F	G	Available
A	11	13	17	14	250
B	16	18	14	10	300
C	21	24	13	10	400
Demand	200	225	275	250.	

Sol: Given that

	D	E	F	G	Available	Penalty
A	11	13	17	14	250	2
B	16	18	14	10	300	4
C	21	24	13	10	400	3
Demand	200	225	275	250	950 / 950	
Penalty	5	5	1	0		

$\Sigma a_i = \Sigma b_j$. The given transportation problem is balanced.

→ ①

	D	E	F	G	Available	Penalty
A	200 | 11	13	17	14	~~250~~ 50	2
B	16	18	14	10	300	4
C	21	24	13	10	400	3
Demand	~~200~~	225	275	250	950 / 950	
penalty	$\boxed{5}$	5	1	0		

→ ②

	E	F	G	Available	Penalty
~~A~~	50 | ~~13~~	~~17~~	~~14~~	50	~~2~~ 1
B	18	14	10	300	4
C	24	13	10	400	3
Demand	~~225~~ 175	275	250	750 / 750	
Penalty	$\boxed{5}$	1	0		

⇒

	E	F	G	Available	Penalty
B	175 | ~~18~~	14	10	~~300~~ 125	4
C	~~24~~	13	10	400	3
Demand	~~175~~	275	250	700 / 700	
Penalty	$\boxed{6}$	1	0		

$\Rightarrow$

	F	G	Available	Penalty
~~B~~	~~14~~ 125 | ~~10~~		~~125~~	$\boxed{4}$
C	13	10	400	3
Demand	275	~~250~~ 125	525 \ 525	
Penalty	1	0		

$\Rightarrow$

	F	~~G~~	Available	Penalty
C	13 125 | ~~10~~		~~400~~ 275	3
Demand	275	~~125~~		

$\Rightarrow$

	F	
C 275	13	275
	275	

$\Rightarrow$

	~~F~~	G	Ava	Pen
C	275 | ~~13~~	10	~~400~~ 125	3
Dem	~~275~~	125		
Pen	13	10		

$\Rightarrow$

	G	Avai	Pen
C	125 | 10	~~125~~	10
Dem	~~125~~	125/125	
Penalty	10		

$\Rightarrow$ Min Cost $Z = 275\times13 + 10\times125 + 125\times10 + 175\times18 + 50\times13 + 200\times11$

$= 12075$ □

2Q) Use Penalty Method and find an IBFS

	D_1	D_2	D_3	Available	Penaltly
O_1	6	8	4	6	2
O_2	4	9	3	10	1
~~O_3~~	~~1~~	15 | ~~2~~	~~6~~	~~15~~	~~1~~
O_4	5	7	2	4	3
Requirement	14	~~16~~ 1	5	35 \ 35	
Penalty	3	5	1		

→ Assignment Problems :- An assignment problem is a particular Case of transportation problem in which a number of jobs. In that Operations or assigned to equal number of persons (Operators), where each person performs only one job. The Objective is to maximize Overall profit or minimize overall Cost for a given assignment schedule.

→ Mathematical Formulation of an Assignment Problem

Consider an assignment problem of assigning Small n jobs to n persons. Let C_{ij} = The Cost incurred in assigning i^{th} job to j^{th} person. $x_{ij} = \begin{cases} 1, \text{ if the } i^{th} \\ 0, \text{ Otherwise.} \end{cases}$

Operation is assigned to j^{th} Operator

∴ The assignment problem is Simply the following L.P.P

Minimize $Z = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} \cdot C_{ij}$ Subject $\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1 \text{ to } n$

and $\sum_{j=1}^{n} x_{ij} = 1$, $i = 1$ to n with $x_{ij} = 1$ or 0.

Q A firm plane to begin production of 3 new products. They Own three plants and wish to assign 1 new plant. The Unit Cost of producing i at plant j is C_{ij}. As given by the following matrix. Find the assignment that minimize the total Unit Cost

$$\text{Product} \quad \overset{\text{Plant}}{\begin{pmatrix} 10 & 8 & 12 \\ 18 & 6 & 14 \\ 6 & 4 & 2 \end{pmatrix}}$$

Sol:- Given that,

let

$$\begin{array}{c|ccc|} & A & B & C \\ \hline 1 & 10 & 18 & 12 \\ 2 & 18 & 6 & 14 \\ 3 & 6 & 4 & 2 \\ \hline \end{array}$$

Step 1 :- Subtracting the Smallest element of each row from every element of the Corresponding row, to get

$$\begin{array}{c|ccc|} & A & B & C \\ \hline 1 & 2 & 0 & 4 \\ 2 & 12 & 0 & 8 \\ 3 & 4 & 2 & 0 \\ \hline \end{array}$$ is a Square matrix

Step 2 :- Subtracting the Smallest element of each Column In reducing matrix, from every element of the Corresponding Column to get

$$\begin{array}{c|ccc|} & A & B & C \\ \hline 1 & 0 & 0 & 4 \\ 2 & 10 & 0 & 8 \\ 3 & 2 & 2 & 0 \\ \hline \end{array}$$

Step 3 : Starting with the first row we rectangle □ [i.e, make assignment] a single zero if any and Cross [x] all other zeros in the Column. So marked, to get

$$\begin{array}{c|ccc} & A & B & C \\ \hline 1 & 0 & \cancel{0} & 4 \\ 2 & 10 & \boxed{0} & 8 \\ 3 & 2 & 2 & \boxed{0} \end{array}$$

Step 4 : Starting with the first Column, we rectangle □ [i.e make assignment] a single zero, If any, and Cross [x] all other zeros in the ~~Column~~ row. So marked, to get

$$\begin{array}{c|ccc} & A & B & C \\ \hline 1 & \boxed{0} & \cancel{0} & 4 \\ 2 & 10 & \boxed{0} & 8 \\ 3 & 2 & 2 & \boxed{0} \end{array}$$

Now, Since each row and each Column has One and only one assignment, an Optimum (Solution) is reached. The Optimum assignment is

assignment
$1 \longrightarrow A$
$2 \longrightarrow B$
$3 \longrightarrow C$

The minimum assignments ~~sed~~ Scheduled $= 10+6+2 = 18$□

Q :- Solve the following assignment problem by using ~~Gen~~ Hungarian method

Task	Men 1	2	3
I	9	26	15
II	13	27	6
III	35	20	15
IV	18	30	20

Sol:- Given that is not a Square matrix first we need to Convert it into Square matrix by adding Column with 0 Cost.

$$\begin{array}{c|cccc} & & \text{Men} & & \\ \text{Task} & 1 & 2 & 3 & 4 \\ \hline I & 9 & 26 & 15 & 0 \\ II & 13 & 27 & 6 & 0 \\ III & 35 & 20 & 15 & 0 \\ IV & 18 & 30 & 20 & 0 \end{array}$$

$\Rightarrow$

$$\begin{array}{c|cccc} & & \text{Men} & & \\ \text{Task} & 1 & 2 & 3 & 4 \\ \hline I & 9 & 26 & 15 & 0 \\ II & 13 & 27 & 6 & 0 \\ III & 35 & 20 & 15 & 0 \\ IV & 18 & 30 & 20 & 0 \end{array}$$

$\Rightarrow$

$$\begin{array}{c|cccc} & & \text{Men} & & \\ \text{Task} & 1 & 2 & 3 & 4 \\ \hline I & 0 & 6 & 9 & 0 \\ II & 4 & 7 & 0 & 0 \\ III & 26 & 0 & 9 & 0 \\ IV & 9 & 10 & 14 & 0 \end{array}$$

$\Rightarrow$

$$\begin{array}{c|cccc} & & \text{Men} & & \\ \text{Task} & 1 & 2 & 3 & 4 \\ \hline I & 0 & 6 & 9 & \cancel{0} \\ II & 4 & 7 & 0 & \cancel{0} \\ III & 26 & 0 & 9 & \cancel{0} \\ IV & 9 & 10 & 14 & \boxed{0} \end{array}$$

$\Rightarrow$

$$\begin{array}{c|cccc} & & \text{Men} & & \\ \text{Task} & 1 & 2 & 3 & 4 \\ \hline I & \boxed{0} & 6 & 9 & \cancel{0} \\ II & 4 & 7 & \boxed{0} & \cancel{0} \\ III & 26 & \boxed{0} & 9 & \cancel{0} \\ IV & 9 & 10 & 14 & \boxed{0} \end{array}$$

$I \rightarrow 1$

$II \rightarrow 3$ $\quad IV \rightarrow 4$

$III \rightarrow 2$

$\Rightarrow 9 + 6 + 20 + 0 = 35$

Q. A departmental head has four subordinates, and four tasks to be performed. The subordinates differ in efficiency, and the tasks differ in their intrinsic difficulties. His estimate of the time each man would take to perform each task is given in the matrix below.

Tasks	Men E	F	G	H
A	18	26	17	11
B	13	28	14	26
C	38	19	18	15
D	19	26	24	10

How should the tasks be allocated, one to a man, so as to minimize the total man hours.

Sol: Given that

Task	E	F	G	H
A	18	26	17	11
B	13	28	14	26
C	38	19	18	15
D	19	26	24	10

is a Square matrix

Subtracting the smallest element of each row from every element of corresponding row, we get

Task	E	F	G	H
A	7	15	6	0
B	0	15	1	13
C	23	4	3	0
D	9	16	14	0

In reducing matrix, Subtracting the smallest element of each ~~row~~ Column from every element of corresponding Column, we get

→

Task	E	F	G	H
A	7	11	5	0
B	0	11	0	13
C	23	0	2	0
D	9	12	13	0

Starting with the row we n rectangle. that is we ~~make~~ a single zero, If any and Cross X all other zeros. in the Column. So marked, we get

⟹

Task	E	F	G	H
A	7	11	5	$\boxed{0}$
B	$\boxed{0}$	11	0	13
C	23	$\boxed{0}$	2	~~0~~
D	9	12	13	~~0~~

• Starting with first Column, We n rectangle □, a single zero, If any and Cross all other zeros in the row, So marked we get.

⇒

Task	E	F	G	H
A	7	11	5	$\boxed{0}$
B	$\boxed{0}$	11	~~0~~	13
C	23	$\boxed{0}$	2	~~0~~
D	9	12	13	~~0~~

→ We observe each row and each Column does not

Since

has only one assignment, go to next step

One and

Step 5 :-

(i) The fourth row does not have any assignment we tick [✓] these row

ii) Now, there is a zero in the fourth Column of the ticked row. So we tick [✓] fourth Column

(iii) Further there is an assignment in the ticked row of the ticked Column. So we tick first row.

iv) Draw the straight lines through all Unticked rows and ticked Columns. We get

$$
\begin{array}{c|cccc|c}
 & E & F & G & H & \\
A & 7 & 11 & 5 & \boxed{0} & \checkmark \\
B & \boxed{0} & 11 & \cancel{0} & 13 & \\
C & 23 & \boxed{0} & 2 & \cancel{0} & \\
D & 9 & 12 & 13 & \cancel{0} & \checkmark \\
 & & & & \checkmark &
\end{array}
$$

<u>Step 6</u> : In Step 5 we observe that the Minimum no. of lines So drawn is 3, which is less than the Order of the Cost matrix Indicating that the Current assignment is not Optimum.

To increase the minimum no. of lines, We generate new zeros in the modified matrix.

<u>Step 7</u> :- The Smallest element not Covered by the lines is 5. Subtracting this element from all Uncovered elements and adding the Same in all the elements lying at the intersection of the lines, We get

	E	F	G	H
A	2	6	0	0
B	0	11	0	18
C	23	0	2	5
D	4	7	8	0

<u>Step 8</u> :- Repeat Step 3 & 4., We get :

	E	F	G	H
A	2	6	0	0
B	0	11	0	18
C	23	$\boxed{0}$	2	5
D	4	7	8	$\boxed{0}$

	E	F	G	H
A	2	6	$\boxed{0}$	✗
B	$\boxed{0}$	11	✗	18
C	23	$\boxed{0}$	2	5
D	4	7	8	$\boxed{0}$

Since we observe that each row and each column has only ~~and~~ one assignment an Optimum Solution is one and reached. The Optimum assignment is

$$A \rightarrow G$$
$$B \rightarrow E$$
$$C \rightarrow F$$
$$D \rightarrow H$$

$\therefore$ The Minimum Man hours Total assignment Schedule $= 17 + 13 + 19 + 10$
$= 59$ hrs.

Note :- If the assignment problem is in the form of maximization, the first we convert it minimization by using, by substracting each element from the highest element of the matrix, then we can solve. Therefore the minimization of the resulting matrix is the same as the maximization of Original matrix.

Eg:- Solve the following assignment problem, to find the maximum total expected Sales.

Areas

$$\text{Salesman}\quad \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} I & II & III & IV \end{array} \\ \begin{pmatrix} 42 & 35 & 28 & 21 \\ 30 & 25 & 20 & 15 \\ 30 & 25 & 20 & 15 \\ 24 & 20 & 16 & 12 \end{pmatrix} \end{array}$$

Sol: The given assignment problem is a maximization, and we convert minimization, by subtracting every element of the cost matrix from the highest element. we get

	I	II	III	IV
A	0	7	14	21
B	12	17	22	27
C	12	17	22	27
D	18	22	26	30

is a square matrix

Step 1 :- Subtracting the smallest element of each row from every element of the corresponding row, we get

	I	II	III	IV
A	0	7	14	21
B	0	5	10	15
C	0	5	10	15
D	0	4	8	12

Step 2 :- Subtracting the smallest element of each column from every element of the corresponding column we get

	I	II	III	IV
A	0	3	6	9
B	0	1	2	3
C	0	1	2	3
D	0	0	0	0

Step 3 :- Starting with first row we rectangle a single zero, If any and cross x all other zeros in the column, So marked we get.

	I	II	III	IV
A	$\cancel{\boxed{0}}$	3	6	9
B	$\cancel{0}$	1	2	3
C	$\cancel{0}$	1	2	3
D	$\cancel{0}$	0	0	0

(4)→ Starting with first Column we n rectangle a single zero. If any and Cross all other zeros in the row. So marked. We get

	I	II	III	IV
A	$\boxed{0}$	3	6	9
B	$\cancel{0}$	1	2	3
C	$\cancel{0}$	1	2	3
D	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$

(5)→ Since We observe each row and ~~every~~ each Column doesnot have one ~~only~~ and only one assignment, go to next step

Step 5 :-

(i) The third and ~~fourth~~ Second row doesnot have any assignment we tick these rows.

(ii) Now there is a zero in the ~~fourth~~ first and ~~third~~ Column of the ticked row.

So we tick third and fourth Columns.

(iii) Further there is an assignment in the first row of the ticked Column, we tick first row

(iv) Draw a straight line of all Unticked rows and ticked Columns we get

	I	II	III	IV	
A	$\boxed{0}$	3	6	7	✓
B	0	1	2	3	✓
C	0	1	2	3	✓
D	$\cancel{0}$	$\boxed{0}$	0	0	

Step 6 : The smallest element not covered by the line is 1. Subtracting these element from all the uncovered elements and adding the same to all the elements lying in the intersection of the lines and remaining are unaltered, we get

	I	II	III	IV
A	0	2	5	8
B	0	0	1	2
C	0	0	1	2
D	1	0	0	0

Step 7 :- Repeat steps 3 and 4, we get.

	I	II	III	IV
A	$\boxed{0}$	2	5	8
B	$\cancel{0}$	$\boxed{0}$	1	2
C	$\cancel{0}$	$\cancel{0}$	1	2
D	1	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$

Therefore the Optimum Solution is not reached. Now

Repeat 5 steps and 6. We get

	I	II	III	IV	
A	$\boxed{0}$	2	5	8	✓
B	$\cancel{0}$	$\boxed{0}$	1	2	✓
C	$\cancel{0}$	$\cancel{0}$	1	2	✓
D	1	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	
	✓	✓			

$$\Rightarrow \begin{array}{c|cccc} & \mathrm{I} & \mathrm{II} & \mathrm{III} & \mathrm{IV} \\ \hline A & 0 & 2 & 4 & \cancel{7} \\ B & 0 & 0 & 1 & \cancel{1} \\ C & 0 & 0 & 1 & \cancel{1} \\ D & 2 & 1 & 0 & 0 \end{array}$$

Repeat 3 and 4

$$\Rightarrow \begin{array}{c|cccc} & \mathrm{I} & \mathrm{II} & \mathrm{III} & \mathrm{IV} \\ \hline A & \boxed{0} & 2 & 4 & \cancel{7} \\ B & \cancel{0} & \boxed{0} & \cancel{0} & \cancel{1} \\ C & \cancel{0} & \cancel{0} & \boxed{0} & \cancel{1} \\ D & 2 & 1 & 0 & \boxed{0} \end{array}$$

Since We assigned arbitary.

$\therefore$ Therefore the assignment schedule is

$A \rightarrow \mathrm{I}$

$B \rightarrow \mathrm{II}$

$C \rightarrow \mathrm{III}$

$D \rightarrow \mathrm{IV}$

Hence :- Maximum total Cost :-

$42 + 25 + 20 + 12 = 99$ □

Assignment Problem Algorithm : (or)

Hungarian method Algorithm

Step 1: Check whether the Cost matrix is a Square. If not, make it Square by adding Suitable no of dummy rows [or Columns] with zero Cost elements.

Step 2: Identify the Smallest Cost elements in each row of the Cost matrix Subtract, these Smallest element from each element in that row. Then there shall be atleast one zero in each row.

Step 3:- In reduced Cost matrix, Identify the Smallest Cost elements in each Columns. Subtract these Smallest element from each element in that ~~row~~ Column. Then there is atleast one zero in each Column.

Step 4:- Examine the rows successively untill a row with exactly one zero is found. End rectangle is zero and Cross all other zeros in its Column. If there are more than one zero, then donot touch that row and go to next row.

Step 5:- Repeat the procedure for the Columns or the reduced Cost matrix. If there is no Single zero In any row or Column of the reduced matrix. Arbitarly Select and n rectangle a Single zero in that Column and Cross all other zeros in its row.

Step 6:- If Each Column and each row has one and only one assignment, the Optimum Solution is reached otherwise, go to next step

Step 7 : (i) Tick the rows in which assignment has not be made

(ii) Tick the Columns which have zeros in the ticked rows.

(iii) Tick the rows which has assignments in the ticked Columns

iv) Draw the straight lines through all Unticked rows and ticked Columns.

Step 8 : Find the Smallest element not Covered by the lines. Subtract these from all Uncovered elements and add the Same at intersection of the lines. Other elements Covered by the lines unchanged.

Step 9 :- Go to step 3 and step 4 and repeat the procedure untill an Optimum Solution is reached.

UNIT III

SEQUENCING PROBLEMS

We determine an appropriate order [Sequence] for a Series of jobs to be done on a finite number of facilities In Some pre assigned order, So as to optimize the total involved time [Cost]

Notations : M_{ij} - Processing time for job i on machine j
X_{ij} - Ideal time on machine j from the end of $(i-1)^{th}$ job to the starting of i^{th} job
T - total elapsed time for processing for all the jobs including ideal time if any.

Problems with n jobs and two machines [or] Johnson's method :-
The iterative procedure for determining the Optimal Sequence for n jobs and two machines M_1 and M_2 in the order of $M_1 M_2$ Can be explained as follows.

Step 1 : Examine the processing time for i^{th} job [i = 1 to n] on machine m_1 and m_2 and Select the minimum of these. That is minimum of Min. $\{M_{i1}, M_{i2}\}$.
Let these minimum occur for some i = k

Step 2 : If the Smallest processing time is for the machine M_1, to k^{th} job first and place it at the begining of the Sequence.
If it is for the machine m_2, do the k^{th} job in the last and place it at the end of the Sequence.

Step 3 : When there is a tie in selecting the minimum ;
processing time the following situations arised
i) If the equal minimum values occur, one for each
machine. Then place the job in the machine M_1 first
and job in the machine M_2 last in the sequence.
ii) If the equal minimum values occur, only for
machine m_1, select the job with larger processing
time in M_2 and place it first in the job
sequence

iii) If the equal minimum values occur, only for
machine m_2, select the job with larger processing
time in m_1 and place it the job sequence lastly.

Step 4 :- Cross of the jobs already assigned.
If all the jobs have been assigned, go to next
step. Otherwise repeat step 1, 2 and 3

Step 5 :- Calculate the processing time for each job on
M_1. This time can be calculated as follows.
The finishing time of i^{th} job in a sequence on M_1 =
[The finishing time of $(i-1)^{th}$ job in a sequence on
M_1] + [The processing time of i^{th} job on M_1]
The starting time for the first job on M_1 is 0.

Step 6 :- Calculate the starting time and finishing time
of each job on M_2 as follows :-
(i) The starting time of first job in the sequence on
M_2 = The finishing time of first job on M_1

(ii) The finishing time of i^{th} job in a Sequence on M_2 = Max {[The finishing time of i^{th} job in a Sequence on M_1], [The starting time of i^{th} job on M_2]} + [The processing time of i^{th} job on M_2]

(iii) The finishing time of $i+1^{th}$ job in a Sequence on M_2 = (Maximum of) Max. {(The finishing time of $(i+1)^{th}$ job in a Sequence on M_1], [The finishing time of i^{th} job in a Sequence on M_2]}

Step 7 :- Calculate the ideal time for machines M_1 and M_2 as follows.

i) The ideal time for machine M_1 = [The finishing time of n^{th} job on M_2] - [The finishing time of n^{th} job on M_1]

ii) The ideal time for Machine M_2 = [The finishing time of first job on M_1] + $\sum_{i=2}^{n}$ [[The starting time of i^{th} job in a Sequence on M_2] - [The finishing time of $(i-1)^{th}$ job in a Sequence on M_2]]

Eg :- In a factory, There are six jobs to perform, each of which should go through two machines A and B, in the order A B. The Processing timings (hrs). For the jobs are given here. You are requested to determine the Sequence for performing the jobs that would minimize that the total elapsed time T. What is the Value of T.

Job :	J_1	J_2	J_3	J_4	J_5	J_6
Machine A :	1	3	8	5	6	3
Machine B :	5	6	3	2	2	10

Sol: Given that

Job	J_1	J_2	J_3	J_4	J_5	J_6
Machine A	1	3	8	5	6	3
Machine B	5	6	3	2	2	10

→ The Smallest processing time in the given problem is 1 on machine A. So perform job J_1 in the begining of the Sequence [order]

J_1					

→ Next the minimum processing time in the reduced problem is 2. which Corresponds to J_4 and J_5 both on machine B. So the Corresponding processing time on J_5 on Machine A is larger than the Corresponding processing time of J_4 on Machine A. Then J_5 will be processed in the last and J_4 will be penultimate, We get the Updated job Sequence is

J_1				J_4	J_5

→ Now there is a tie among three jobs The Smallest processing time in the reduced problem. is Corresponds to J_2 and J_6 on Machine A, and also to J_3 on Machine B. So The Corresponding processing time of J_6 on machine B is larger than the Corresponding processing time of J_2 on machine B. Hence J_6 will be processed. first and next ~~to~~ J_2 and also J_3 should be Placed last in last in the sequence.

Then the Updated Job Sequence is

$\boxed{J_1 \mid J_6 \mid J_2 \mid J_3 \mid J_4 \mid J_5}$

The total elapsed time and the ideal times on machines A and B are Calculated as follows.

	Machine A		Machine B		Idle Time	
Job	In	Out	In	Out	on A	on B
J_1	0	1	1	6	–	1
J_6	1	4	6	16	–	–
J_2	4	7	16	22	–	–
J_3	7	15	22	25	–	–
J_4	15	20	25	27	–	–
J_5	20	26	27	29	3	–

Therefore the total elapsed time (T) = 29 hrs.

The ideal time on Machine A = 3 hrs

" " " on " B = 1 hr.

The above information Can be represented in the graph, Called Gantt chart

2) A book binder has one printing press and one binding machine, and the manuscripts of a number of different books. The time required to perform the printing and binding operations for each book are shown below. Determine the order in which books should be processed inorder to minimize the total time required to turn out all doubles. to all the books.

→

Book	1	2	3	4	5	6
Printing time	30	120	50	20	90	110
Binding	80	100	90	60	30	10

[PM]

Sol :- Given that, Let Machine M_1 = Printing time, M_2 = Binding time (BM) The Smallest processing time in the given problem is 10. on Machine B(2). So perform Book 6 in the ending of the Sequence.

					6

→ Now the minimum processing time in the reduced problem is 20 on machine 1. So perform the corresponding book 4 in the begining of the Sequence

4					6

→ The minimum processing time in the reduced problem is 30 tie occurs on machine M_1 and M_2. So, perform first book in the begining of the Sequence and 5^{th} book in the ending of the Sequence

4	1			5	6

→ Next the minimum processing time in the reduced problem is 50 on machine M_1. So perform the third book in the starting of the sequence and also last book 2 at the ending of the remaining card.

4	1	3	2	5	6

can be processed next.

The total elapsed time and idle time on machines

	Machine A1		Machine B		Idle Time	
Jobs	In	Out	In	Out	On 1	On B
4	0	20	20	80	–	20
1	20	50	80	160	–	–
3	50	160	160	250	–	–
2	160	290	250	350	–	–
5	290	310	350	380	–	–
6	310	420	380+40 = 420	430	10	40

∴ The total elapsed time = 430 hrs

The ideal time on printing press on machine $M_1 = 10$

" " " " Binding machine $M_2 = 60$

The above information can be represented in the graph called Gantt shot

Problems with n jobs & 3 machines

Suppose let there are n jobs and 3 machines M_1, M_2 & M_3. Each job is processed all the three machines in order M_1, M_2, M_3. For this problem, as it is johnson's method can not be applied. If one of the following 2 conditions satisfied, then we can apply sequence position & go to next step, otherwise this method fails:

i) $\text{Min.}\{M_{i_1}\} \geq \text{Max.}\{M_{i_2}\}$

ii) $\text{Min.}\{M_{i_3}\} \geq \text{Max.}\{M_{i_2}\}$

Now n jobs & 3 machines problem can be reduced n job & 2 machine problem.

We define 2 artificial machines M_{iG} and M_{iH} such that the processing time for this machines

$M_{iG} = M_{i_1} + M_{i_2}$

$M_{iH} = M_{i_2} + M_{i_3}$

Now this n jobs & 2 machines problems, for this johnson's rule can be applied directly. After finding the sequence the total elapsed time & ID times can be ~~tabulated~~ calculated in tabular form.

Q Determine optimal sequence of jobs that minimizes the total ellapsed time based on the following information. Processing time on machines is given in ~~hours~~ hours and passing is not allowed.

Job	A	B	C	D	E	F	G
machine-1:	3	8	7	4	9	8	7
machine-2:	4	3	2	5	1	4	3
machine-3:	6	7	5	11	5	6	12

Job	A	B	C	D	E	F	G
m1	3	8	7	4	9	8	7
m2	4	3	2	5	1	4	3
m3	6	7	5	11	5	6	12

In the given problem, there are 7 Jobs, each of which is to be processed through three machine m_1, m_2, m_3, in order m_1, m_2, m_3.

If one of the following 2 conditions are satisfied, then n jobs and 3 machines problem can be reduced into n jobs and 2 machines problem, then we can apply Johnson's method.

(i) $\min.\{m_{i1}\} \geq \max.\{m_{i2}\}$

(ii) $\min.\{m_{i3}\} \geq \max.\{m_{i2}\}$

$\therefore \min.\{m_{i1}\} = 3$

$\max.\{m_{i2}\} = 5$

$\min.\{m_{i3}\} = 5$

Hence second condition is satisfied

Job	A	B	C	D	E	F	G
machine-G = M1+M2	7	11	9	9	10	12	10
machine-H = M2+M3	10	10	7	16	6	10	15

The minimum processing time is 6 on machine H then the corresponding Job can be processed last in the sequence, we set

						E

Now, the minimum processing time is 7, the ... then the corresponding job A can be processed first in the sequence and job C is last in the sequence

A					C	E

, we set.

Now the minimum processing time is 9 on machine G, then the corresponding job D can be processed first in the sequence, we set.

A	D				C	E

Now the minimum processing time is 10 on the occurs, then the corresponding job G can be processed first in the sequence and the processing time on machine G of Job F is larger than job B, then job F can be processed last in the sequence and job B shall be (next) penultimate

A	D	G	B	F	C	E

→ Find the sequence that minimizes the total time required in performing the following jobs on three machines in the order A. B. C

Processing Time (in hrs) on	Job 1	2	3	4	5	6
Machine - A	8	3	7	2	5	1
Machine - B	3	4	5	2	1	6
Machine - C	8	7	6	9	10	9

Sol :- Given that

Table

In the given problem there are six jobs and each of which is to be processed through three Machines A, B and C in order.

If one of the following two Conditions are Satisfied, then n jobs and three machines problems Can be reduced into n jobs and 2 machine problem. Then We Can apply johnsons method

i) $\text{Min}\cdot\{M_{i_1}\} \geq \text{Max}\{M_{i_2}\}$

ii) $\text{Min}.\ \{M_{i_3}\} \geq \text{Max}.\ \{M_{i_2}\}$

$\therefore$ $\text{Min of } \{M_{i_1}\} = 1$

$\text{Max of } \{M_{i_2}\} = 6$

$\text{Min of } \{M_{i_3}\} = 6$

Hence two Conditions are Satisfied.

Job	1	2	3	4	5	6
Machine G = $M_1 + M_2$	11	7	12	4	6	7
Machine H = $M_2 + M_3$	11	11	11	11	11	15

The minimum processing time is 4 on Machine G then the Corresponding job 4 Can be ~~placed~~ processed first in the Sequence, We get

4					

Now the minimum processing time is 6 on machine G then the Corresponding job 5 can be processed first in the Sequence, We get

4	5				

Now the minimum processing time is 7, tie occurs, then the Corresponding job 6 is placed at the first on the Sequence and job 2 shall be penultimate

4	5	6	2		

Now the minimum processing time is 11 tie occurs, the Corresponding job 1 is processed at the starting of the Sequence and the Corresponding job 3 is processed at the last of the Sequence

4	5	6	2	1	3

The total elapsed time and idle time on MA, MB, MC

Job	Machines M1 In	M1 Out	M2 In	M2 Out	M3 In	M3 Out	Idle time M1	M2	M3
4	0	2	2	4	4	13	-	2	4
5	2	7	4+3 = 7	8	13	23	-	3	-
6	7	8	8	14	23	32	-	-	-
2	8	11	14	18	32	39	-	1	-
1	11	19	18+1 = 19	22	39	47		4	-
3	19	26	22+4 = 26	31	47	53	27	22	-

Total elapsed time = 53 hrs.

M_A = 27 hrs

M_B = 32 hrs

M_C = 4 hrs

10

53
31
22

Problems with two jobs and K Machines.

Let there are two jobs and K machines $M_1, M_2 \ldots M_K$ and each job processed through all K machines. The order of each, of the two jobs are known. The ordering may not be Same for the jobs. The expected processing time for all the machines are known. Now the objective is to determine an optimal Sequence. Such that the total elapsed time is minimum.

To find the optimal Sequence, we use graphical method the procedure is as follows.

Steps :-

① Mark the processing time for 2 jobs on the Corresponding access in the given technological order.

② Draw two perpendicular lines, The horizontal axis represents processing time on job 1 and the Vertical axis represents processing time on job 2.

③ Construct Various jobs by pairing the Same machines from starting point to ending point.

④ Draw the line starting from horizon to ending point by moving horizontally, Vertically and diagonally making an angle on 45°. with horizontal. Moving to right means that job 1 is underprocess while job 2 is idle and moving upward means that job 2 is underprocess while Job one is Idle and moving diagonal means that both jobs are processing.

(5) An Optimal path is 1 Variable that minimizes the idle time for job 1 [horizontal Movement]. Similarly an Optimal path is 1 that minimizes path 2 [Vertical movement]. Choose such a path on which diagonal movement maximum.

(6) The total elapsed time is obtained by adding idle time for either job to the processing time for that job.

Eg:- Choose / Use graphical method to minimize the time added to process the following jobs on the machines shown, i.e for each machine find the graph which should be done first. Also Calculate total elapsed time to Complete both the jobs.

Job-1 :	Sequence	A	B	C	D	E
	Time	3	4	2	6	2

Job 2	Sequence	B	C	A	D	E
	Time	5	4	3	2	6

Sol:- Given that

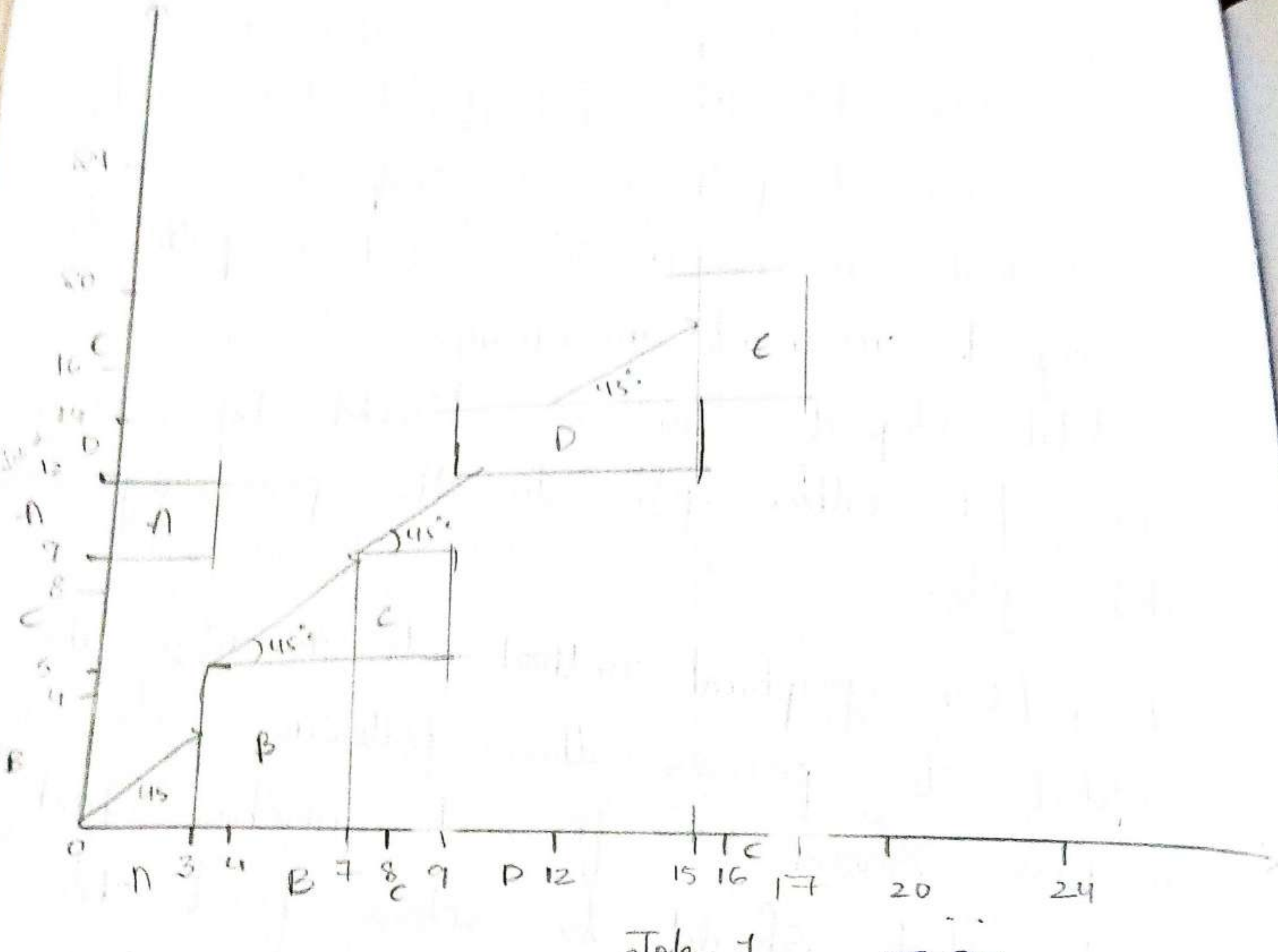
Now

Job-1

Sequence	Time	Cummulative
A	3	3
B	4	7
C	2	9
D	6	15
E	2	17

Job 2

Sequence	Time	Cummulative
B	5	5
C	4	9
A	3	12
D	2	14
E	6	20



T=20.

∴ Therefore the total elapsed time = 20

UNIT - 4

GAME THEORY

Many Practical Problems require decision making in a Competitive Situation, where there are two or more opposing parties with Conflicting interests and where the action of one depends upon one taken by the opponent.

Eg ① Candidates for an election, Advertising and marketing Campaigns by Competitive business firms, etc have their Conflicting interests.

In a Competitive Situation, the courses of alternatives for each competitor may be either finite or infinite. A Competitive Situation is called a game; it has the following properties.

→ There are a finite no. of Competitors [Participants] are Called players

→ Each player has a finite no. of strategies [alternatives] available to give

→ A play of the game takes place when each player employees his strategy

→ Every game results in an outcome, that is loss or gain are a draw, usually Called pay off, to Supplier.

Two person, Zero Sum game :

Games having the zero - Sum Character that algebraic sum of gains and lossess of all the players is zero, Called Zero Sum games.

Zero Sum games with two players are Called two person zero Sum games or rectangular games. In this Case, the loss [or gain] of one player is exactly equal to the gain [or loss] of the other.

The gains resulting from a two person zero Sum game Can be represented in the matrix form Called payoff matrix.

Eg: We Consider a two person coin tossing a game. Each player tosses an unbiased los Coin Simultaneously. Player B pay ₹7/- If (Head, Head) Occurs and ₹4 (Tail, Tail) Occurs Otherwise play A pay ₹3 to B. Therefore player A ~~play off~~ Pay off matrix Can be represented as follows Player B

		H	T
Player A	H	7	-3
	T	-3	4

→ The Maximin – Minmax Principle.

To explain the Maximin–Minmax principle for the Selection of the optimal strategies by the two players. We assume that both the players are Conservative i.e while player A believes that employee his strategy A_1 and his opponent knows that he is going to employ A_1. and Similarly the player B believes so about player A while employing his moves

→ Saddle Point and Value of the game :

A Saddle point ~~and~~ of a payoff matrix is that position. In the payoff matrix where the maximum of row minima coincides with the minimum of Column maxima. The payoff at the starting point is Called the Value of the game and is equal to the maximin and Minimax Values of the game.

Note → The Saddle point, Value of the game may not be unique

- The Value of the game is denoted by V.
- The Maximin Value of the game denoted by $\underline{V}$ (that equal) to lower Value.
- The Minmax Value of the game denoted by $\overline{V}$ = (that equal) to Upper Value
- If lower Value = Upper Value = 0, then the game is said to be a fair game.
- If lower Value = Value of game = Upper Value = then the game is said to be strictly determinable.

Strictly determinable → Optimum strategies

fair game → not " "

① Sol:

i) Given that

		Player B		
		B_1	B_2	Row Minimum
Player A	A_1	5	0	0
	A_2	0	2	0
Column Maximum		5	2	

$\therefore$ The Maximum row of Minima $= 0 = \underline{V}$ = Maximum Value

The Minimum of Column maxima $= 2 = \bar{V}$ = Minimax Value

Hence * & + both doesn't exist for any element,

then there is no Saddle point.

$\underline{V} = 0 \neq \bar{V} = 2$ then the game is not fair.

(ii) Given that

		Player B		
		B		Row Minimum
		B_1	B_2	
Player A	A_1	0	2	0
	A_2	-1	4	-1
Column Maximum		0	4	

The maximum of row minima $= 0 = \underline{V}$ = Maximin Value

and the minimum of Column maxima $= 0 = \bar{V}$ = Minimax Value

Hence the given problem * & + both exists

for a element 0:

$\therefore$ The Saddle point $= 0$ = Value of the game (v).

i) $\underline{V} = 0 = \bar{V}$, then the given game is said to be

fair game.

ii) $\underline{V} = V = \overline{V}$, then the given game is said to be strictly determinable thus the optimal strategy for players A & B are $S_0 = (A_1, B_1)$

8) For the game with way of matrix

Player B

$$\begin{array}{cc} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 2 & 6 \\ -2 & \mu \end{bmatrix} \end{array}$$

Player A

(i) Show that G is strictly determinable whatever μ may be.

(ii) Determine the Value of the game

Sol:

Given that

		Player B		Row Minima
		B_1	B_2	
Player A	A_1	$+\ 2\ *$	$+\ 6$	2
	A_2	$-2\ *$	μ	-2
	Column ~~Minima~~ Maxima	2	6	

$\therefore$ Maximum of row minima $= 2 = \underline{v} =$ maximin Value.

minimum of Column maxima $= 2 = \overline{v} =$ minimax Value

Hence both ~~and~~ star and plus exist for the element 2.

Thus the Value of the game $v = 2$

Therefore $\underline{v} = v = \overline{v}$, Then the game is strictly determinable. and the optimum strategies for the players A and B is $S_0 = [A_1 \quad B_1]$

(i) Given that

Game is Strictly determinable for any Value of μ

(ii) The Value of the game $= 2$

For the game with pay off matrix

$$\begin{array}{cc} & \text{Player A} \\ \text{Player B} & \begin{bmatrix} -1 & 2 & -2 \\ 6 & 4 & -6 \end{bmatrix} \end{array}$$

Determine the best strategies for player A and B and also the value of the game for M. Is this game first ~~fair~~ (i) fair (ii) strictly determinable

Sol:- Given that

		Player A			
		A_1	A_2	A_3	Row minima
Player B	B_1	-1	2	-2^*	-2
	B_2	6	4	-6	-6
Column Maxima		6	4	-2	

Therefore :- The maximin value $(\underline{v}) = -2$

The minmax value $(\overline{v}) = -2$

Hence the both star and plus are both allocated for the same element $= -2$.

Therefore the starting point $= -2 = \underset{(v)}{}$ The value of the game

And $\underline{v} = v = \overline{v}$, Then the game is strictly determinable and not fair

The best strategy for players B and A

$$S_0 = [B_1 \ A_3]$$

Note :

→ The saddle point is also Called Equilibrium point Saddle.

Games Without Saddle point : Mixed Strategies

To determining the maximum of row minima and the minima of Column maxima are two different operations, there is no reason to expect that there should always lead to unique way of position [Saddle point]

In all such Cases to Solve games both the players must determine the Optimal mixer of Strategies to find a Saddle point. The Optimal mixer strategy for each player may be determined by assigning to each Strategy, its probability of being Choosen. So, the determined strategies are Called Mixed Strategies because they are ~~probabilitistic~~ probabilistic ~~sive~~ Combination of available choices of ~~Stategy~~ Strategy. The Value of game obtained by using mixed strategy that represents player A can expect to Win and least ~~whitch~~ which player B Can lost.

Solution of 2×2 rectangular games:

A game without saddle point can be Solved by Various Solutions methods. In most of the Situations, the given rectangular game Can be reduced to a much Smaller 2×2 Game. Therefore

It is to determine formulae for Optimal Strategies and the Value of the game. In the Case of 2×2 game,

For any 2×2, 2 person 0 sum game without any saddle point having the pay off matrix for player - A

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & a_{11} & a_{12} \\ A_2 & a_{21} & a_{22} \end{array}$$

Therefore the Optimum mixed Strategies for players A and B are

$$S_A = \begin{bmatrix} A_1 & A_2 \\ P_1 & P_2 \end{bmatrix}$$

and

$$S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_3 \end{bmatrix}$$

where $$P_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$\Longrightarrow P_2 = 1 - P_1 \qquad \therefore P_1 + q_2 = 1$$

and $$q_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$\Rightarrow q_2 = 1 - q_1 \qquad \therefore q_1 + q_2 = 1$$

The Value of the game $(v) = \dfrac{(a_{11} \times a_{22}) - (a_{12} \times a_{2}}{(a_{11} + a_{22}) - (a_{12} + }$

1? the game with the following way of matrix determine Optimum strategies the Value of the game

$$\begin{array}{cc} & Q \\ P & \begin{pmatrix} 5 & 1 \\ 3 & 4 \end{pmatrix} \end{array}$$

Sol: Given that

		Q: q_1	Q: q_2	Row minima
P	P_1	5^+	1^*	1
	P_2	3^*	4^+	3
Column maxima		5	4	

Therefore maximum Value $[\underline{v}] = 3$

minimum Value $[\overline{v}] = 4$

Since both star and plus does not exist for any element. Hence there is no ~~starting~~ saddle point

Now, the Optimum strategies and the Value of the game are Calculated as follows.

Now, Given that

$$\begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ Say}$$

Therefore
$$P_1 = \frac{a_{22}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})}$$

$\therefore P_1 + P_2 = 1$

$$= \frac{4-3}{[5+4]-[1+3]} = \frac{1}{[9-4} = \frac{1}{5}$$

$\Rightarrow$

$P_2 = 1 - 0.2$

$P_2 = 0.8$

$$q_1 = \frac{a_{22}-a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{4-1}{9-4} = \frac{3}{5} = 0.6$$

$q_2 = 1 - q_1 \Rightarrow 1 - 0.6 = 0.4.$

The Value of game =

$$\frac{[a_{11}\times a_{22}] - [a_{12}\times a_{21}]}{(a_{11}+a_{22}) - [a_{12}+a_{21}]}$$

$$\frac{[5\times 4] - [1\times 3]}{9 - 4}$$

$$= \frac{20+3}{\cancel{5}\,5} = \cancel{23/5} = \cancel{4}$$

$17/\cancel{5}5 \square$

Hence the Optimum Strategies for players p and q are

$$S_P = \begin{pmatrix} P_1 & P_2 \\ P_1 & P_2 \end{pmatrix} \begin{bmatrix} P_1 & P_2 \\ \frac{1}{5} & 4/5 \end{bmatrix}$$

and $S_Q = \begin{bmatrix} Q_1 & Q_2 \\ q_1 & q_2 \end{bmatrix} = \begin{bmatrix} Q_1 & Q_2 \\ \frac{3}{5} & 2/5 \end{bmatrix}$

$v = 17/5 \square$

· Consider a modified form of matching biased points in game problem. In the Matching player is played ₹8.00 stating both heads and having both ties. Non matching player is paid ₹3. when the two Coins do not match. Given the choices of B the matching or Non-matching player, which one would you choose and what would be your strategy

		Non Matching Player	
		H	T
Matching player	H	8	-3
	T	-3	1

Given that

$$\begin{array}{ccccc} & & \text{Non Matching Player} & & \\ & & H & T & \text{Row Minima} \\ \text{Matching Player} & H & \begin{pmatrix} 8 & -3 \\ -3 & 1 \end{pmatrix} & & \begin{array}{c} -3 \\ -3 \end{array} \\ & T & & & \\ \text{Column Maxima} & & 8 & 1 & \end{array}$$

Maximum Value $(\underline{v}) = -3$

Minimum Value $(\overline{v}) = 1$

$$\begin{bmatrix} 8 & -3 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{say}$$

$$P_1 = \frac{a_{22} - a_{21}}{(a_{11}+a_{22}) - (a_{12}+a_{21})} = \frac{1-(-3)}{(8+1)-[-3+(-3)]}$$

$$= \frac{4}{9-[-6]} = \frac{4}{9+6} = \frac{4}{15} = 0.26$$

$$P_2 = 1 - P_1$$
$$= 1 - 0.26 = 0.74$$

$$q_1 = \frac{a_{22} - a_{12}}{(a_{11}+a_{22}) - (a_{12}+a_{21})} = \frac{1-(-3)}{15} = \frac{4}{15} = 0.26$$

$$q_2 = 1 - q_1 = 0.74$$

$$v = \frac{[a_{11} \cdot a_{22}] - [a_{12} \cdot a_{21}]}{[a_{11} + a_{22}] - [a_{12} + a_{21}]} = \frac{[8 \cdot 1] - [-3 \cdot -3]}{9+6}$$

$$= \frac{8-9}{15} = \frac{-1}{15}$$

$$S_{\text{matching}} = \begin{array}{cc} H & T \end{array} \begin{bmatrix} \frac{11}{15} & \frac{4}{15} \end{bmatrix} \qquad S_{\text{non-matching}} \begin{array}{cc} H & T \end{array} \begin{bmatrix} \frac{11}{15} & \frac{4}{15} \end{bmatrix}$$

$$v = -\frac{1}{15}$$

⟹ Graphline Method

The Graphline method is useful for the game, if the way of matrix each of the size $2\times n$ or $m\times 2$ Then we reduce 2×2 game and we can solve

Eg:- Solve the following game graphically

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} 1 & 3 & 11 \\ 8 & 5 & 2 \end{bmatrix} \end{array}$$

Sol:- Given that

		Player B B_1	B_2	B_3	Row minima
Player A	A_1	1	3	11	1
	A_2	8	5	2	2
Column maxima		8	5	11	

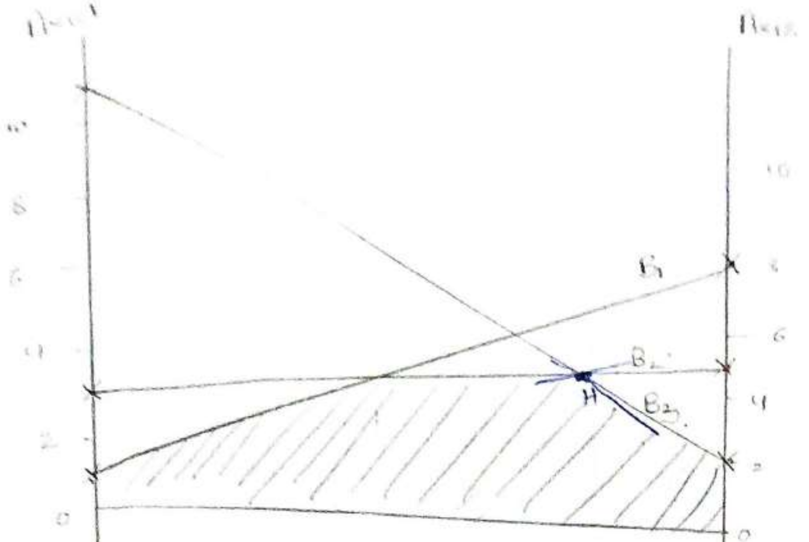
$2\times n$

Maximum Value $[\underline{v}] = 2$ MaxMin Value.

Minimum Value $[\overline{v}] = 5$ Minimax Value

Since both star and plus does not exist for the same element ~~i~~ saddle

Hence there is no ~~starting~~ element. Now the given 2×3 way off matrix can we reduced into 2×2 game by using graphing method we can solve.



We observe the above graph in the lower boundary, the highest point [H] intersect with the lines B_2 and B_3, Therefore the reduced 2×2 is

$$\Rightarrow \begin{array}{c|cc} & B_2 & B_3 \\ \hline A_1 & 3 & 11 \\ A_2 & 5 & 2 \end{array} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ say}$$

The Optimum strategies for players A and B and the Value of game

$$P_A = \frac{a_{22}-a_{21}}{(a_{11}+a_{22}) - [a_{12}+a_{21}]} = \frac{2-5}{[3+2]-[11+5]} = \frac{2-5}{5-16} = \frac{-3}{-11} = 0.2727$$

$$P_2 = 1 - P_A$$

$$\frac{14-3}{11} \qquad = 1 - \frac{3}{11} = 0.7273$$

$$Q_B = \frac{a_{22}-a_{12}}{(a_{11}+a_{22}) - [a_{12}+a_{21}]} = \frac{2-11}{-11} = \frac{-9}{-11} = \frac{9}{11} = 0.8181$$

$$Q_2 = 1 - Q_1 = 1 - \frac{9}{11} = \frac{2}{11}$$

$$(v) = \frac{(a_{11}\cdot a_{22}) - (a_{12}\cdot a_{21})}{(a_{11}+a_{22}) - (a_{12}+a_{21})} = \frac{6-55}{5-16} = \frac{-49}{-11}$$

$\frac{49}{11}$ □

Hence The Optimum Strategies for players A and B are

$$S_A = \begin{bmatrix} A_1 & A_2 \\ B_1 & B_2 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ \frac{3}{11} & \frac{8}{11} \end{bmatrix}$$

$$S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ 0 & q_1 & q_2 \end{bmatrix} = \begin{bmatrix} B_1 & B_2 & B_3 \\ 0 & \frac{9}{11} & \frac{2}{11} \end{bmatrix}$$

→ Solve the following A with the pay off matrix

$$\begin{array}{cc} & A \\ B & \begin{bmatrix} 1 & 3 & -3 & 7 \\ 2 & 5 & 4 & -6 \end{bmatrix} \end{array}$$

Given that

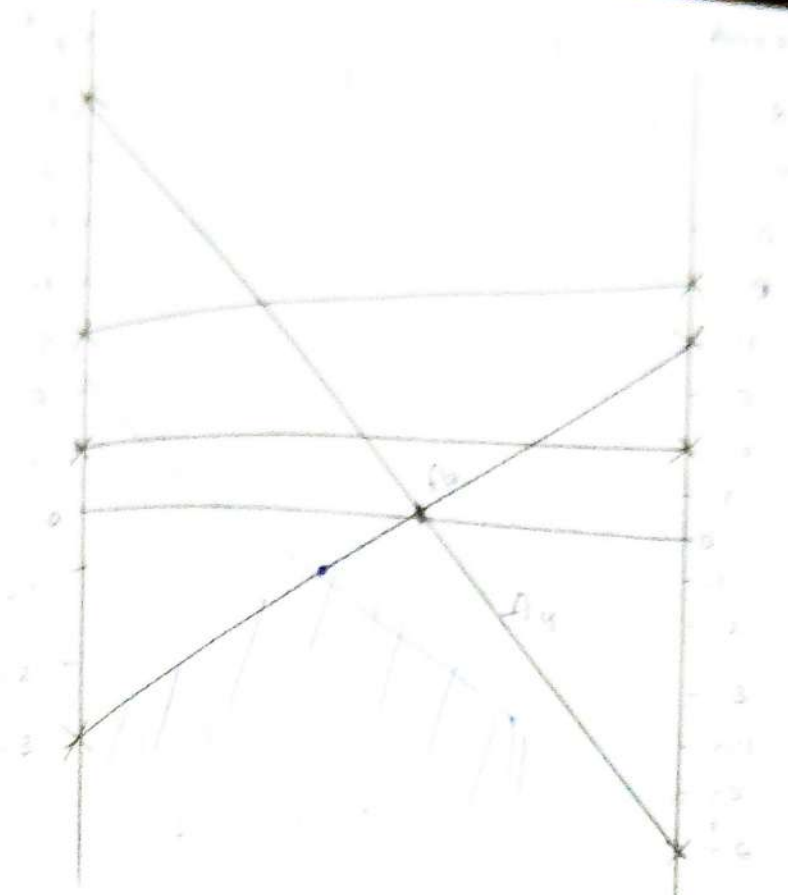
$$\begin{array}{ccc} & A & \text{Row Minima} \\ & \begin{array}{cccc} & & A_3 & \end{array} & \\ B & \begin{bmatrix} 1 & 3 & {}^{+}-3 & {}^{*}7 \\ {}^{*}2 & {}^{*}5 & {}^{*}4 & {}^{+}-6 \end{bmatrix} & \begin{array}{c} -3 \\ -6 \end{array} \\ \text{Column Maxima} & \begin{array}{cccc} 2 & 5 & 4 & 7 \end{array} & \end{array}$$

Maximum Value $[\underline{v}] = -3$

Minimum Value $(\bar{v}) = 2$

A 2×4

$2\times m$



	A_3	A_4
B_1	-3	7
B_2	4	-6

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$P_1 = \frac{a_{22} - a_{21}}{(a_{11}+a_{22}) - (a_{12}+a_{21})} = \frac{-6+4}{(-3-6)-(7+4)} = \frac{-10}{-9-11} = \frac{-10}{-20} = \frac{1}{2}$$

$$\frac{1}{2}$$

$$Q_1 = \frac{a_{22} - a_{12}}{(a_{11}+a_{22}) - (a_{12}+a_{21})} = \frac{-6-7}{-20} \quad \frac{-13}{-20} = 0.65$$

$$Q_2 = \frac{7}{20}$$

$$(v) = \frac{(a_{11}\cdot a_{22}) - (a_{12}\cdot a_{21})}{(a_{11}+a_{22}) - (a_{12}+a_{21})} = \frac{(-3\times-6)-(7\cdot 4)}{-9-11} \quad \frac{18-28}{-20} = \frac{-10}{-20} = \frac{1}{2}$$

Hence the Optimum Strategies

$$S_A = \begin{bmatrix} A_1 & A_2 \\ B_1 & B_2 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ q_1 & q_2 & q_3 & q_4 \end{bmatrix} \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 0 & \frac{13}{20} & \frac{7}{20} \end{bmatrix}$$

$$v = 1/2$$

Note: If the pay off matrix of $m \times 2$ game is also Counted in the Same Way expect that the lowest point of upper boundary.

Eg: Obtain the Optimal Strategies for both persons and the Value of the game for 2 person zero sum game, whose pay off matrix is as follows :

$$\begin{array}{cc|cc|c} & & \multicolumn{2}{c|}{\text{Player } B} & \\ & & B_1 & B_2 & \text{Row Minima} \\ \hline \text{Player } A & A_1 & 1 & -3^{*} & -3 \\ & A_2 & 3^{*} & 5 & 3 \\ & A_3 & -1^{*} & 6^{+} & -1 \\ & A_4 & 4^{+} & 1^{*} & 1 \\ & A_5 & 2^{*} & 2 & 2 \\ & A_6 & -5^{*} & 0 & -5 \\ \hline & \text{Column Maxima} & 4 & 6 & \end{array}$$

$\boxed{m \times 2}$

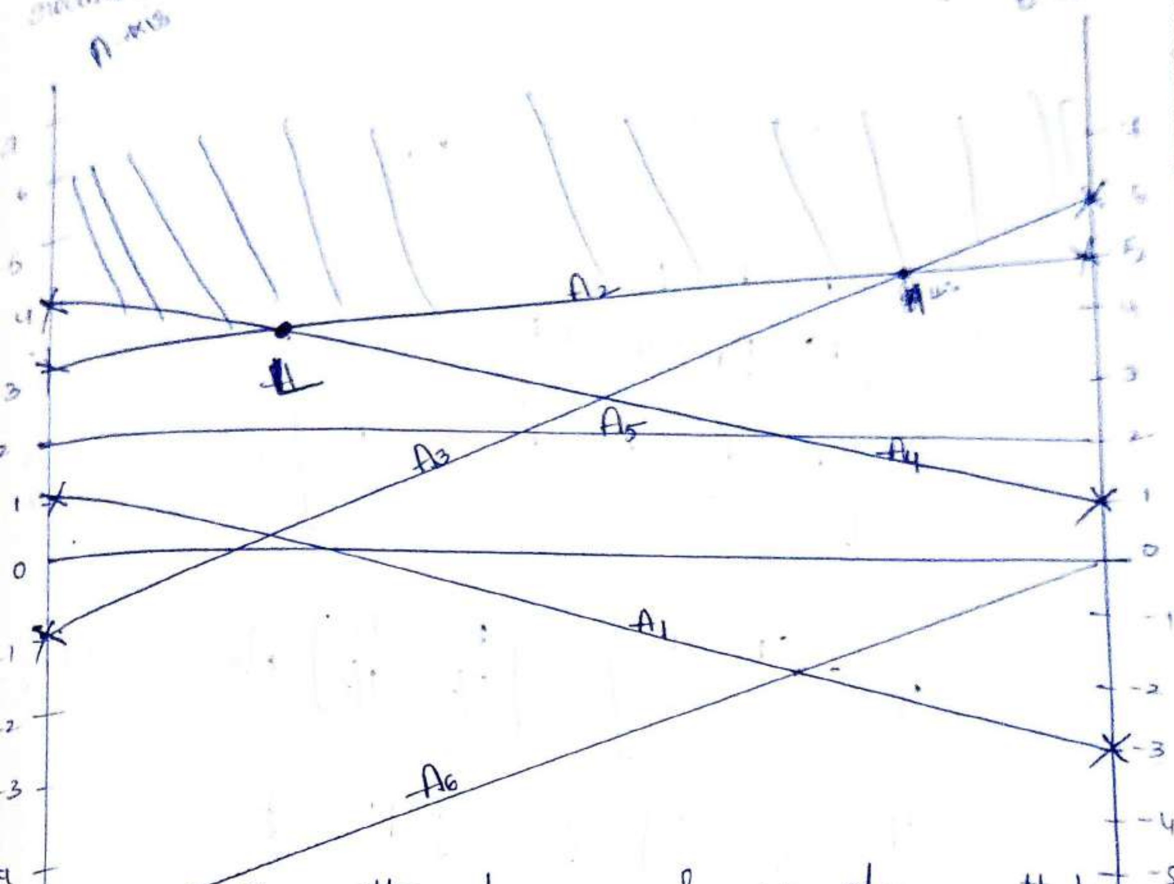
Given that

Maximum Value 3

Minimum Value 4

Since both star and plus does not exist for the same element. Hence there is no ~~starting~~ Saddle element.

Now the given 6×2 pay off matrix can be reduced.



From the above graph we observe that in the upper boundary the lowest point (L) intersecting with the lines A_2 and A_4.

The reduced 2×2 is

$$\begin{array}{c|cc} & B_1 & B_2 \\ A_2 & 3 & 5 \\ A_4 & 4 & 1 \end{array} \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ say}$$

The Optimum Strategies for Players A and B.

$$P_1 = \frac{a_{22} - a_{21}}{(a_{11}+a_{22}) - (a_{12}+a_{21})} = \frac{1-4}{(3+1) - (5+4)} = \frac{-3}{4-9} = \boxed{\frac{3}{5}}$$

$$\boxed{P_2 = \frac{2}{5}}$$

$$Q_1 = \frac{a_{22} - a_{12}}{(a_{11}+a_{22}) - (a_{12}+a_{21})} = \frac{1-5}{-5} = \frac{-4}{-5}$$

$$\boxed{Q_1 = 4/5} \qquad \boxed{Q_2 = \frac{1}{5}}$$

The Value of the game

$$v = \frac{(a_{11} \cdot a_{22}) - (a_{12} \cdot a_{21})}{(a_{11}+a_{22}) - (a_{12}+a_{21})}$$

$$= \frac{(3 - 20)}{4 - 9} = \frac{-17}{-5} = \frac{17}{5}$$

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ 0 & P_1 & 0 & P_2 & 0 & 0 \end{bmatrix}$$

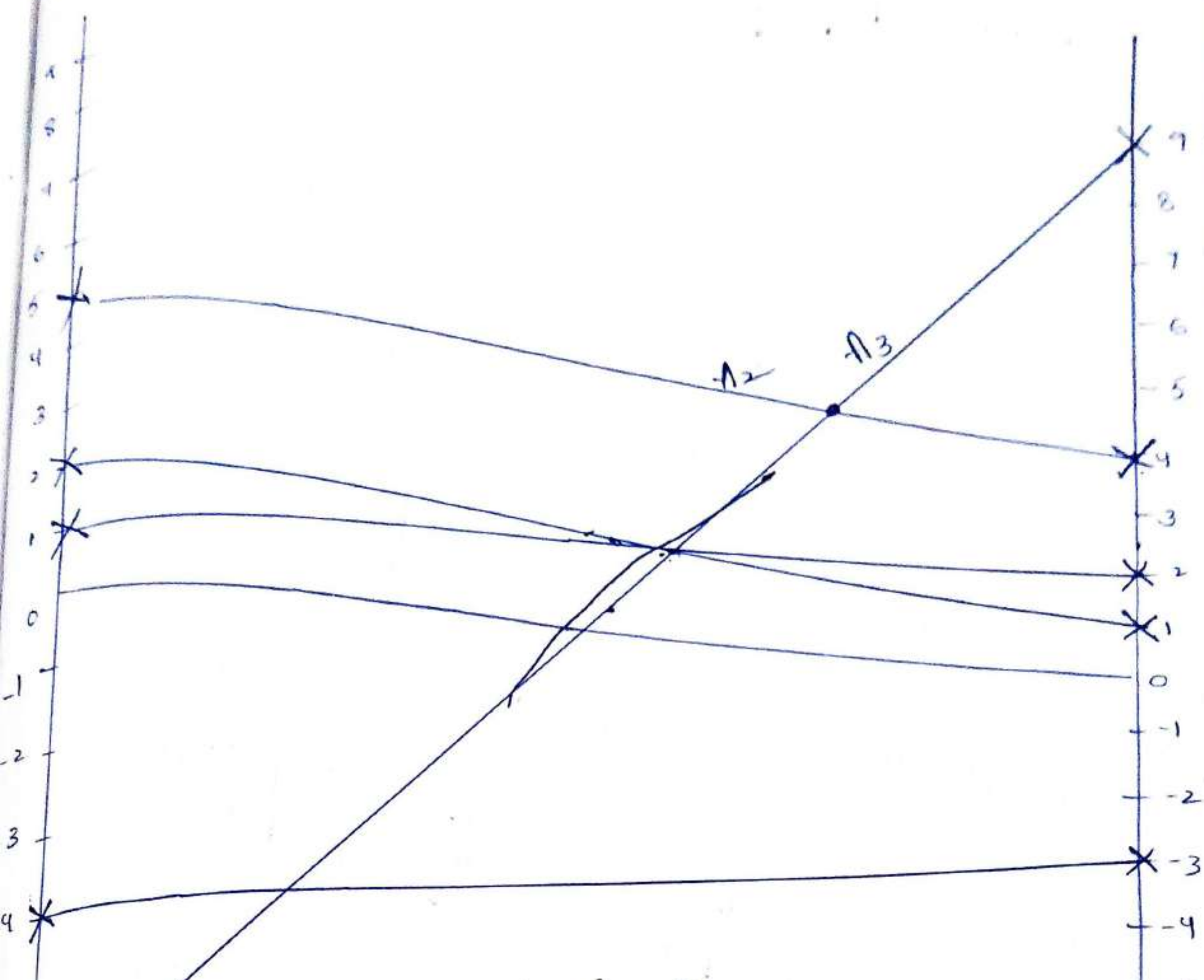
$$\begin{bmatrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ 0 & \frac{3}{5} & 0 & \frac{2}{5} & 0 & 0 \end{bmatrix}$$

$$S_B = \begin{bmatrix} B_1 & B_2 \\ Q_1 & Q_2 \end{bmatrix} \quad \begin{bmatrix} B_1 & B_2 \\ 4/5 & 1/5 \end{bmatrix} \text{ Also } v = \frac{17}{5}$$

— Solve the following game graphically.

Sol: Given that

$$\begin{array}{c|c|c} & B & \text{Row minimum} \\ A & \begin{bmatrix} 1 & 2 \\ 5 & 4 \\ -7 & 9 \\ -4 & -3 \\ 2 & 1 \end{bmatrix} & \begin{matrix} 1 \\ 4 \\ -7 \\ -4 \\ 1 \end{matrix} \\ \text{Column Maximum} & \begin{matrix} 5 & 9 \end{matrix} & \end{array}$$



	B_1	B_2
A_2	5	4
A_3	-7	9

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$P_1 = \frac{a_{22} - a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{9+7}{14+3} \quad P_1 \ \frac{16}{17} = \frac{1}{17} = P_2$$

$$Q_1 = \frac{a_{22} - a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})} \quad \frac{9-4}{17} \quad Q_1 \ \frac{5}{17} = Q_2 = 12/17$$

$$v = \frac{[a_{11} \cdot a_{22}] - [a_{12} \cdot a_{21}]}{[a_{11}+a_{22}] - [a_{12}+a_{21}]} = \frac{[5 \cdot 9] - [4 \cdot -7]}{14+3} = \frac{45+28}{17}$$

$$= 73/17 \ \square$$

→ Dominance Property.

If the given game with pay off matrix of the order m×n then we apply the dominance property, and it can be reduced 2×n (or) m×2 game then we can apply graphing method and we can solve.

→ The general rules for dominance.

1. If all the elements of a row, say k^{th}, are less than or equal to the corresponding elements of any other row, say r^{th} then k^{th} row is dominated by r^{th} row.

→ If all the elements of a column, say k^{th}, are greater than or equal to the corresponding elements of any other column, say r^{th}, then k^{th} column is dominated by k^{th} row.

→ Dominated rows & columns may be deleted to reduce the size of matrix, as the optimal strategies will remain unaffected.

Eg: As the following 2 person 0 some game stable the pay off is for player A solve the game.

$$\text{Player A}\begin{array}{c} \text{Player B} \\ \begin{pmatrix} 5 & -10 & 9 & 0 \\ 6 & 7 & 8 & 1 \\ 8 & 7 & 15 & 1 \\ 3 & 4 & -1 & 4 \end{pmatrix} \end{array}$$

Sol:

Given that

		Player B				
		B_1	B_2	B_3	B_4	Row Minima
Player A	A_1	5	-10	9	0	-10
	A_2	6	7	8	1	1
	A_3	8	7	15	1	1
	A_4	3	4	-1	4	-1
	Column Maxima	8	7	15	4	

Maximin Value $[\underline{v}] = 1$ Maximin

Minimax Value $[\bar{v}] = 4$

∴ Since both star and plus does not exist for any element in the given payoff matrix then there is no ~~starting~~ Saddled point now the given 4×4 pay off matrix Can be reduced by dominance property.

∴ Given that

$$\begin{matrix} 5 & -10 & 9 & 0 \\ 6 & 7 & 8 & 1 \\ 8 & 7 & 15 & 1 \\ 3 & 4 & -1 & 4 \end{matrix}$$

Row Dominance

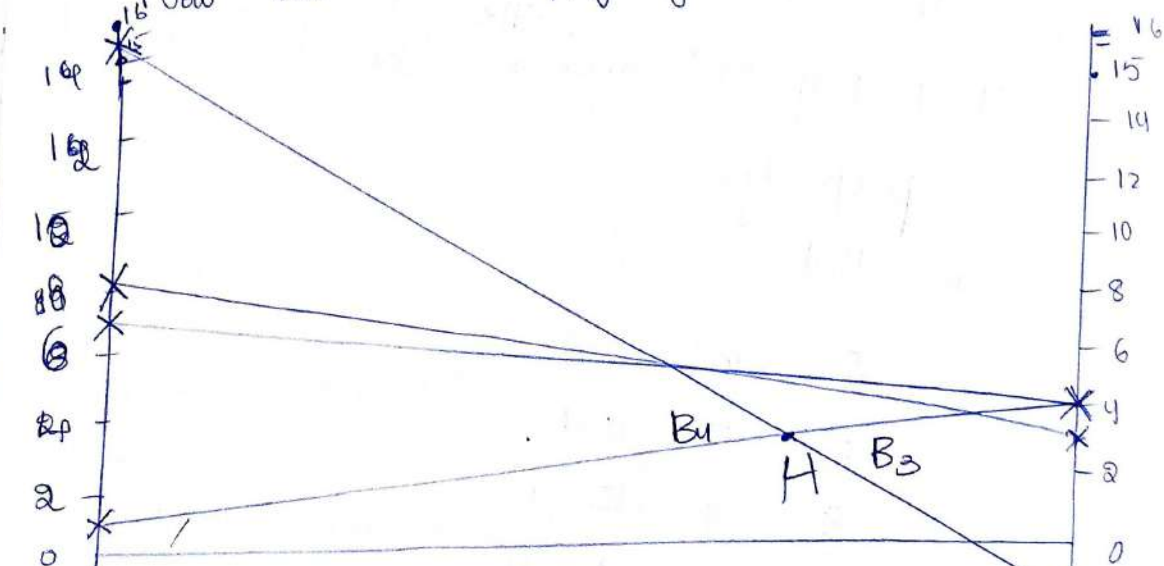
We observe that all the elements of the first row are less than or equal to all the elements of thirdrow, then first row is dominated by thirdrow. Hence first row is deleted. ~~The first~~ All the elements of Second row are less than or equal to the ~~Correspon~~ Corresponding elements of the third row. Hence Second row is dominated by third row. Hence Second row may be deleted

<u>Column Dominance</u>

→ All the elements of any Column are not greater than or equal to the Corresponding elements of any Column. Any Column does not dominate therefore the reduced pay off matrix of the game is $2\times n$.

$$\begin{array}{c|cccc} & B_1 & B_2 & B_3 & B_4 \\ \hline A_3 & 8 & 7 & 15 & 1 \\ A_4 & 3 & 4 & -1 & 4 \end{array}$$

Now We Can apply graphic method



$$\begin{array}{c|cc} & B_3 & B_4 \\ \hline A_3 & 15 & 1 \\ A_4 & -1 & 4 \end{array} \quad \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$P_1 = \frac{a_{22}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{4+1}{19-0} = \frac{5}{19} \qquad P_2 = \frac{14}{19}$$

$$Q_1 = \frac{a_{22}-a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})} \quad \frac{4-1}{19} = \frac{3}{19} \qquad Q_2 = \frac{16}{19}$$

$$v = \frac{[a_{11} \cdot a_{22}] - [a_{12} \cdot a_{21}]}{[a_{11} + a_{22}) - (a_{12} + a_{21}]} = \frac{(15 \times 4) - (-1)}{19} = \frac{61}{19}$$

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & 0 & P_1 & P_2 \end{bmatrix} = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & 0 & \frac{5}{19} & \frac{14}{19} \end{pmatrix}$$

$$\begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 0 & q_1 & q_2 \end{bmatrix} \quad \begin{pmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 0 & \frac{3}{19} & \frac{16}{19} \end{pmatrix}$$

$$v = \frac{61}{19} \square$$

Chapter 5

Project Management

Network : Theory

Networks are diagrams, easily visualized in electrical theory, transportation Systems like roads, railway lines and pipe lines etc

Network Schedulling :-

Network Schedulling is a technique used for planning & Schedulling large projects in the fields of Construction, Maintainence, Purchasing, Computer System installation etc... It is a method of minimizing trouble Spots such as production delays & interuptions by determining Critical factors & co-ordinating Various parts of overall job

There are '2' basic planning & Control techniques that utilize a method / Network to Complete a pre-determined project. These are :

(i) Critical Path Method [CPM]

(ii) Programme evaluation review technique [PERT]

Generally CPM networks are activity Orientation placing the emphasis on the descriptions associated with activities in a network.

The (PERT) networks are generally 'event oriented' emphasis the descriptions associated with the events.

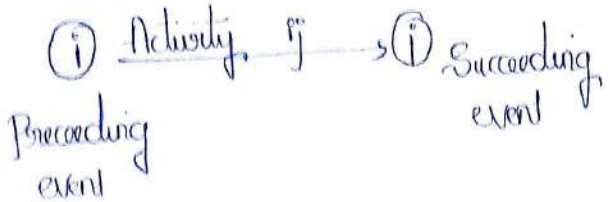
Basic terms :

Event (or) Node : The beginning & end points of an activity, are called events. Event is a point in time & does not consume any resources. It is represented by a numbered circle, the i th event called i th ... event always has a number higher than the event tale. It is also called the j th event.

Activity :

An Activity is a task (or) item of work to be done that consumes time, money (or) other resources. It lies between '2' events called the preceeding & succeding one's. an activity is represented on the network by an arrow with its head indicating the sequence in which the events are to occur.

(i)



(ii)

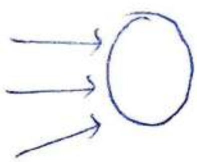
→ Event A contains two Activities 'AB' & 'AC'

(iii)

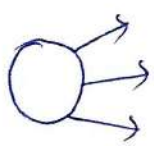
→ Activities (XZ) & (YZ) Contain event 'Z'

Merge & burst events :

It is not necessary for an event to be ending event of only '1' activity as it can be ending event of '2' (or) more activities such event is called Merge event

(iv) 

If the event happens to the begining event of 2 (or) more activities it is called a burst event

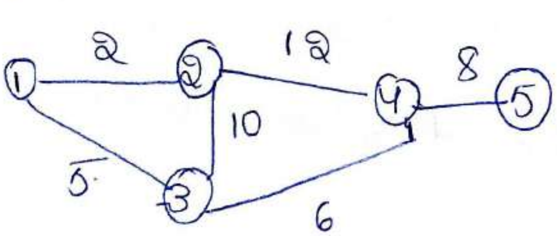
(v) 

→ Draw a network diagram to the following data

Activity	1-2	1-3	2-3	2-4	3-4	4-5
Duration (in dates	2	5	10	12	6	8

Sol :- Given that]

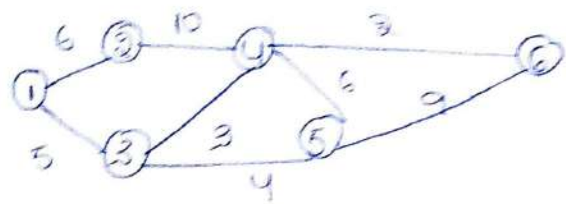
Diagram :-



Draw a network diagram for the following data

Activity	1-2	1-3	2-4	3-4	3-5	4-5	4-6	5-6
Duration (in Dates)	6	5	10	3	4	6	8	9

Sol: Given that



→ Critical Path Method [CPM]:

Steps:

① List all the jobs (&) then draft a network diagram. Each job is indicated by an arrow with the direction of the arrow showing the sequence of jobs. The length of the arrows has no significance. Place the jobs on the diagram ① by ① keeping in mind what precedes & follows each job as well as what jobs can be done simultaneously.

② Consider the jobs times to be deterministic indicate them above the arrow representing the task.

③ Calculate the earliest start time & earliest finish time for each event & write them in the box (□). Calculate the latest start time [LST] & lastest finish time [LFT] & write them in the triangle. (Δ)

④ Tabulate Various times that is activity normal times, earliest times & latest times on the mark (EST) & (LFT) on the arrow diagram

⑤ Determine the total (Slack) for each activity by taking difference between (EST) & (LFT)

⑥ Identify the Critical activities & Connect them with the begining node & the ending node in the network diagram by double line arrows, this is in CPM.

⑦ Calculate the total Project Duration.

Note ① The EST = ES_i (or) E_i

② EFT = $E_{si} + t_{ij}$ (or) E_j

③ LST = (LS_{ij}) = $LF_{ij} - t_{ij}$ (or) $L_j - t_{ij}$

④ LFT = LF_{ij} (or) L_j

⑤ Total Quote $(TF_{ij}) = LS_{ij} - ES_{ij}$

⑥ Free Quote $(FF_{ij}) = (E_j - E_i) - t_{ij}$.

Forward Fast Calculations :-

In this We estimate the earliest start (ES_i) & earliest finish time (ES_j)

The earliest start time for the event 'i' is given by $E_i = \max\{ES_i + t_{ij}\}$

For any problem EST for first event is taken as '0'. i.e ; $ES_i = 0 = E_1$

Backward Fast Calculations

→ In this, we Calculate the latest finish & the (LST)

→ The LFT for an event 'i' is given by

$L_i = \text{Min.}\ (LF_{ij} - t_{ij})$

where (LF_j = LFT for the event j)

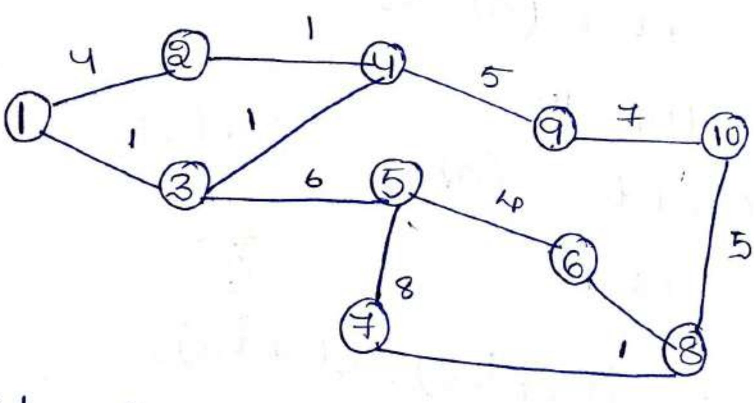
→ T_{ij} = The normal time of the activity

for any problem we take $L_n = E_n$

→ From the following data draw a network diagram

Activity	1-2	1-3	2-4	3-4	3-5	4-9	5-6	5-7	6-8	7-8	8-10	9-10
Normal time (t_{ij}) in days	4	1	1	1	6	5	4	8	1	2	5	7

and also Calculate EST & EFT, LST & LFT, Total Quote & free Quote Critical activities, CP, Total Project Duration



To Calculate EST & EFT

Forward Fast Calculations.

→ In this we estimate the EST & EFT (ES_j & ES_j)

→ The EST for the event I is given by E_i = max. $\{ES_i + t_{ij}\}$

→ We, take the EST for the first event

$E_{S1} = 0 \cdot E_1$

① $ES_2 = E_{S1} + t_{12}$ (or) $E_2 = E_1 + t_{12}$

① $E_{S2} = ES_1 + t_{1,2}$ (or) $E_2 = E_1 + t_{12}$

$= 0 + 4 = 4$

② $= E_{S3} = E_{S1} + t_{13}$ (or) $E_1 + t_{13} = 0 + 1 = 1$

③ $ES_4 = \text{Max} \cdot (ES_2 + t_{24}) + (ES_3 + t_{3,4}) \}$

$E_4 = \text{Max} \cdot \{ (E_2 + t_{2,4}) + (E_3 + t_{34}) \}$

$= \text{Max} \cdot \{ (4+1), (1+1) \}$

$= \text{Max} \cdot \{ 5, 2 \}$

$= 5$

④ $ES_5 = ES_3 + t_{35}$ (or) $E_5 = E_3 + t_{3,5}$

$= 1 + 6 = 7$

⑤ $ES_6 = ES_5 + t_{5,6}$ (or) $E_6 = E_5 + t_{56}$

$= 7 + 4 = 11$

⑥ $ES_7 = ES_5 + t_{57}$ (or) $E_7 = E_5 + t_{57}$

$= 7 + 8 = 15$

⑦ $E_{S8} = \text{Max} \cdot \{ (E_6 + t_{68}), (E_7 + t_{78}) \}$

$= \text{Max} \cdot \{ (11+1), (15+2) \}$

$= (12, 17)$

$= 17$

⑧ $E_{S9} = E_4 + t_{49}$

$E_9 \Rightarrow 5 + 5 = 10$

$$E_{10} = \max \cdot \{(E_8 + t_{8,10}), (E_9 + t_{9,10})$$

$$= \max \cdot \{17 + 5\}, (15, + 7)\}$$

$$= 22$$

We Calculate LFT :-

<u>Backward</u> <u>Fast</u> <u>Calculations</u> :-

→ In this we Calculate the latest finish time [LFT] & (LST)

→ The latest time for i^{th} event is given by

$$L_i = \min \cdot \{LF_j - t_{ij}\}$$

→ For any problem we take the 'LFT' for n^{th} event

$$(L_n) = E_n$$

$$\therefore L_{10} = E_{10} = 22$$

$$\rightarrow L_9 = L_{10} - t_{9,10} = 22 - 7 = 15$$

$$\rightarrow L_8 = L_{10} - t_{8,10} = 22 - 5 = 17$$

$$\rightarrow L_7 = L_8 - t_{7,8} = 17 - 2 = 15$$

$$\rightarrow L_6 = L_8 - t_{6,8} = 17 - 1 = 16$$

$$\rightarrow L_5 = \text{Min} \{(L_7 - t_{5,7}), (L_6 - t_{5,6})\}$$

$$\text{Min} = \{15 - 8), (16 - 4)\}$$

$$\text{Min} = (7, 12)$$

$$\text{Min} = 7$$

$$\rightarrow L_4 = L_9 - t_{4,9}$$

$$= 15 - 5 = 10$$

$$\rightarrow L_3 = \text{Min} \{(L_5 - t_{3,5}), (L_4 - t_{3,4})\}$$

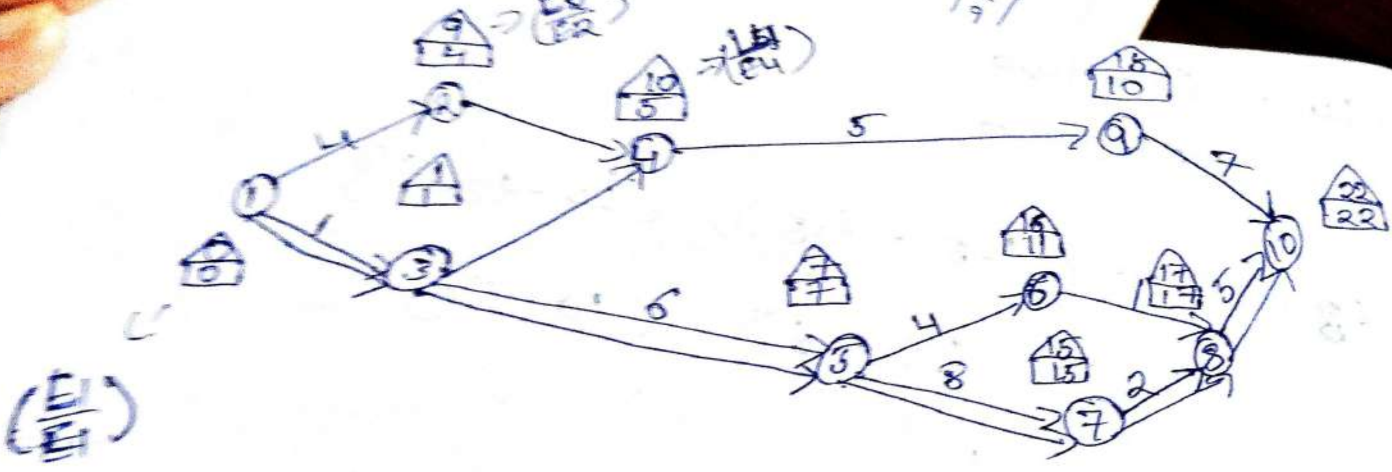
$$= (7 - 6), (10 - 1) = 1$$

$$\rightarrow L_2 = (L_4 - t_{24}]$$

$$= 10 - 1 = 9$$

(ESj=ESi+tij) (LSi=Lj-tij) (Lj)

① Activity	② Normal time (Tij)	③ Earliest start	④=3+2 finish	⑥=⑤-② Latest start	⑤ finish	TF	FF
1-2	4	$E_1=0$	4	(9-4)=5	$L_2=9$	(5-0)=5	(4-0)-4=0
1-3	1	$E_1=0$	1	(1-1)=0	$L_3=1$	(0-0)=$\boxed{0}$	(1-0)-1=0
2-4	1	$E_2=4$	5	(10-1)=9	$L_4=10$	(9-4)=5	(5-4)-1=0
3-4	1	$E_3=1$	2	9	$L_4=10$	8	(2-1)-1=0
3-5	6	$E_3=1$	7	1	$L_5=7$	$\boxed{0}$	(7-1)-6=0
4-9	5	$E_4=5$	10	10	$L_9=15$	5	(10-5)-5=0
5-6	4	7	11	12	16	5	0
5-7	8	7	15	7	15	$\boxed{0}$	0
6-8	1	11	12	16	17	5	0
7-8	2	15	17	15	17	$\boxed{0}$	0
8-10	5	17	22	17	22	$\boxed{0}$	0
9-10	7	10	17	15	22	5	0



The, critical activities are [1-3, 3-5, 5-7, 7-8, 8-10]

The, critical path is [1-3, 3-5, 5-7, 7-8, 8-10]

The, total project duration = 1+6+8+2+5 → Duration of normal time (t_{ij})

= 22 days

PERT [Program Evaluation and Review Technique]

→ Map a list of activities that make up the project including immediate predecessors.

→ Making use of step 1, sketch the required network.

→ Denote the Most likely time by t_m, the optimistic time with t_o and pessimistic time t_p.

→ Using β distribution for the activity duration, the expected time t_e [Normal time], for each activity is calculated by using the formulae

$$t_e = \frac{t_o + 4 t_m + t_p}{6}$$

→ Tabulate various times, i.e Expected and activity times, earliest and latest times and mark them on the arrow diagram. Determine the total flowate and free flowate for each activity.

→ Identify the critical activities and connect them with begining node and the ending node in the network diagram by double line arrows. This gives the critical path and the expected date of confusion of the Project

→ Using the values of t_p and t_o, calculate the variance of each activities time estimates by using $\sigma^2 = \left[\frac{t_p - t_o}{6}\right]^2$

→ Calculate the standard normal deviate

$$Z_o = \frac{\text{Duedate} - \text{Expected date of Completion}}{\sqrt{\text{Project Variance}}}$$

i.e $z = \frac{x - \mu}{\sigma}$

→ Use Standard normal tables to find the Probability $P(z \le z_0)$ of Completing the project within the Scheduled time, where $z \sim N(0,1)$

Eg:- A Small Project is Composed of Seven activities whose time estimates are listed in the table as follows

Activity i j	Estimated duration (Weeks) Optimistic	Most likely	Pessimistic
1 2	1	1	7
1 3	1	4	7
1 4	2	2	8
2 5	1	1	1
3 5	2	5	14
4 6	2	5	8
5 6	3	6	15

→ Draw a project network

→ Find the Expected duration (Normal time) and Variance of each activity

→ Calculate early and late Occurence times for each event. What is the expected project length.

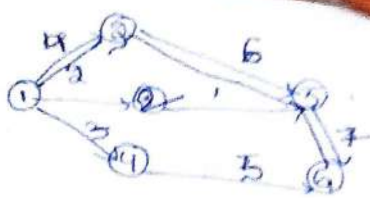
→ Calculate the Variance and Standard deviation of Project length. What is the Probability that the Project will be Completed (i) Atleast 4 Weeks Earlier than expected (ii) No more than four Weeks later than expected.

Sol: Given that

Activity (i-j)	Estimated duration Optimistic (t_0)	most likely (t_m)	Pessimistic (t_p)
1-2	1	1	7
1-3	1	4	8
1-4	2	2	1
2-5	1	1	14
3-5	2	5	8
4-6	2	5	15
5-6	3	6	

Draw network diagram with the help of Activity and normal time

$t_e = \frac{t_0 + 4t_m + t_p}{6} = t_{ij}$ (Normal time)	$\sigma^2 = \left(\frac{t_p - t_0}{6}\right)^2$
① $= \frac{1+4(1)+7}{6} = \frac{1+4+7}{6} = \frac{12}{6} = 2$	$\left(\frac{7-1}{6}\right)^2 = 1$
② $\frac{1+4(4)+7}{6} = 4$	1
③ $\frac{2+4(2)+8}{6} = 3$	1
④ $\frac{1+4(1)+1}{6} = 1$	$\left[\frac{1-1}{6}\right]^2 = 0$
⑤ $\frac{2+5(4)+14}{6} = 6$	$\left[\frac{14-2}{6}\right]^2 = 4$
⑥ $\frac{2+5(4)+8}{6} = 5$	$\left[\frac{8-2}{6}\right]^2 = 1$
$\frac{3+24+15}{6} = 7$	$\left[\frac{15-3}{6}\right]^2 = 4$



→ To Calculate Earliest start time

forward fast Calculations

We know that $E_j = \text{Max}\cdot\{E_i + t_{ij}\}$

Next first $E_1 = 0$

$E_1 = 0$

$E_2 = E_1 + t_{12}$

$= 2$

$E_3 = E_1 + t_{13}$

$= 4$

$E_4 = E_1 + t_{14}$

$0 + 3 = 3$

$E_5 = \text{Max}\cdot\{(E_2 + t_{25}),$

$(E_3 + t_{3,5})$

$= (2 + 1, (4+6)\}$

$= 10$

$E_2 = E_1 + t_{12}$

$= 0 + 4 = 5$

$E_3 = E_1 + t_{13}$

$= 0 + 2 = 2$

$E_4 = E_1 + t_{14}$

$= 0 + 3 = 3$

$E_5 = \text{Max}\cdot\{(E_2 + t_{25}), (E_3 + t_{35})\}$

$= \text{Max}\{(5+6), (2+1)\}$

$= \text{Max}\cdot(11, 3)$

$= 11$

$E_6 = \text{Max}\cdot\{(E_4 + t_{4,6}), (E_5 + t_{5,6})\}$

$\text{Max}\cdot\{(3+5), (11+7)\}$

$= (8, 18)$

18

$E_6 = \text{Max}\{(E_4 + t_{4,6})$

$(E_5 + t_{5,6})$

$= 17$

→ To Calculate latest Finish time

Backward fast Calculations

We know that $L_i = \text{Min}\cdot\{L_j - t_{ij}\}$

We take $L_n = E_n$

$L_6 = E_6$

$= 17$

$L_5 = L_6 - t_{5,6}$ $17 - 7 = 10$

$= 17 - 7 = 10$

$L_4 = L_6 - t_{4,6}$

$= 17 - 5 = 12$

$L_3 = L_5 - t_{3,5}$

$= 10 - 6 = 4$

$L_2 = L_5 - t_{2,5}$

$10 - 1 = 9$

$L_1 = \text{Min} \cdot \{(L_3 - t_{13}), (L_2 - t_{12}), (L_4 - t_{1,4})\}$

$\text{Min} \cdot \{(9-1), (4-4), (12-3)\}$

$= \text{Min}\ (0, \ldots)$

$= 0.$

Activity	Normal Time	Earliest		Latest		TF	FF
		Start	Finish	Start	Finish		
1-2	2	0	2	7	9	7	0
1-3	4	0	4	0	4	$\boxed{0}$	0
1-4	3	0	3	9	12	9	0
2-5	1	2	3	9	10	7	0
3-5	6	4	10	4	10	$\boxed{0}$	0
4-6	5	3	8	12	17	9	0
5-6	7	10	17	10	17	$\boxed{0}$	0