Digital Signal Processing (HM423)

Introduction to DSP:-

Digital Signal Processing (DSP) is the field of engineering and applied mathematics focused on converting, analyzing and manipulating real-world signals like sound, images and sensor data, using digital methods and specified specialized algorithms.

What is DSP?

DSP involves transforming analog sig (continuous sig from the real world) into digital sigs (discrete sequence of no's sampled at intervals), then using computational techniques to enhance, analyze (or) modify these signals.

Core steps in DSP.

* Sig sampling:- Analog signals are sampled to produce discrete-time representations. This involves dividing the sig into equal time intervals and measuring amplitudes.
* Quantization:- Amplitude measurements are rounded to fixed levels, resulting in finite sets of digital values.

* Processing:- Digital algorithms - such as filtering, Fourier transforms and compression - are applied to manipulate sigs for various purposes, including noise removal, feature enhancement, and data extraction.

Applications:-

1. Audio and voice sig processing (Eg: music players, noise cancelling headphones)
2. Image and video processing and compression (Streaming platforms, digital cameras).
* Telecommunication (mobile phones, data txion error correction).
* Biomedical sig Analysis (ECG, medical imaging)

DSP signals in different Domains:

Time / Space domain:- Direct analysis of sampled data over time (or) across spatial domain.

Frequency domain:- Transforms sigs to analyze their spectral content, often using the DFT.

Wavelet Domain:- Breaks sigs into components at different scales for detailed analysis.

What is DSP used for?

* DSP is quite complex, it can solve digital sigs, but also carry out a variety of other functions including filtering, compression and modulation.
* we have to reduce the noise errors in the transmitted sig ADC is used to reduce the noise.
* In the final process with the help of DAC the digital sig is comeback to analog sig. To remove noise reduce errors in that sig. It can be done by Digital signal processing.

Why DSP is important?

DSP is important because of it enables the manipulation and analysis of digital signals to improve their quality, extract useful information, and create new functionalities across many modern technologies.

Key Areas:-

Audio and communication:- Noise cancellation, speech processing, Telecommunications.

Image and Video:- Image enhancement and compression medical imaging.

Control Sims:- Automation and robotics, Autonomous vehicles.

Other Applications:- Scientific research, Entertainment.

Advantages :-

* High Accuracy :-
* Noise immunity
* Flexibility & reprogrammability
* Error correction
* Data storage and transpal
* cost effectiveness. / cheaper.

Drawbacks :-

* Sim complexity
* Power consumption
* Bandwidth limited by sampling rate

Applications :-

* Speech Processing
* Communication
* Biomedical
* Consumer Electronics
* Seismology
* Image processing
* Sensor and Control
* Signal detection and tracking
* Software defined radio.

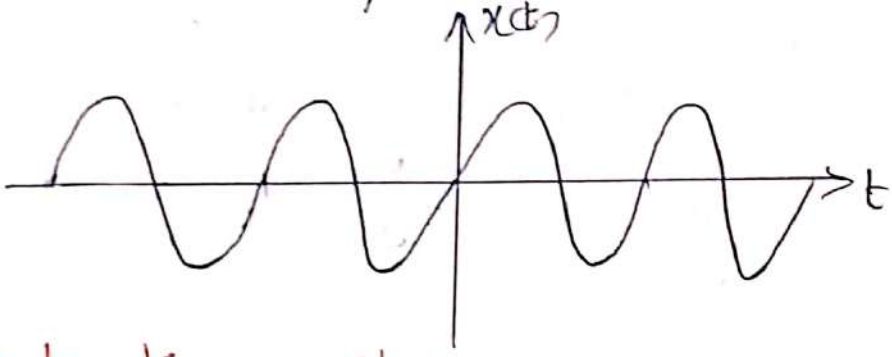
Signal:-

Any physical phenomenon that carries some information is called a signal.

Classification of signals:-

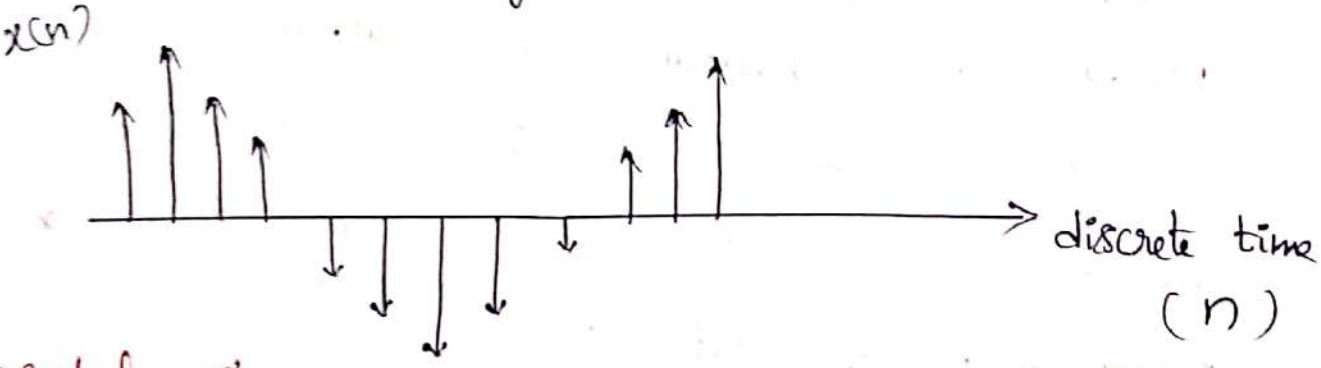
Continuous time Sig (or) Analog Sig:-

A sig is said to be continuous time when it is defined for all instants of time.



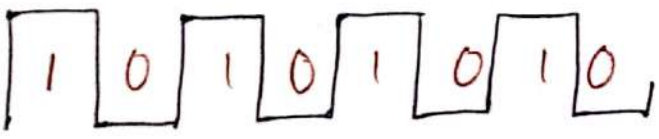
Discrete time sig:-

A sig is said to be discrete, when it is defined at only discrete instants of time.



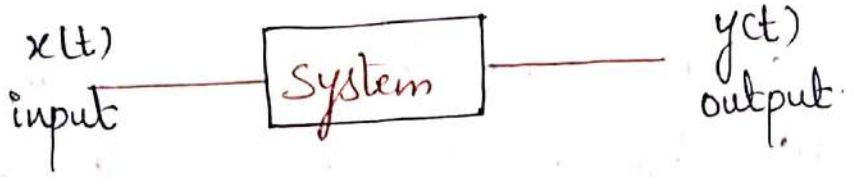
Digital Sig:-

A digital sig is a sig that represents data as a sequence of discrete values.



System:-

A system is defined as a physical device that generates a response (or) an output signal, for a given input sig.



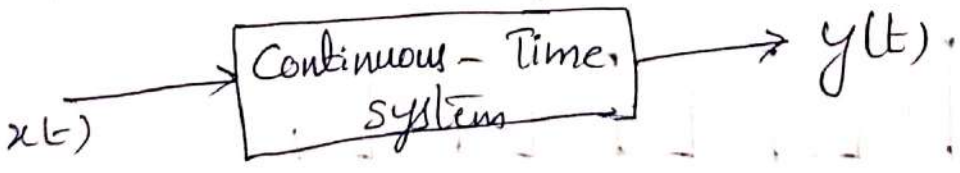
Mathematically the relationship btw the input $x(t)$ and corresponding output $y(t)$ of a system is $y(t) = T[x(t)]$

Classification of systems:-

Continous Time system:-

A Continuous time system is one which operates on a continuous time sig and produces a continuous time output signal. If the input and output of continuous time system are $x(t)$ and $y(t)$, then we say that $x(t)$ is transformed to $y(t)$ i.e.,

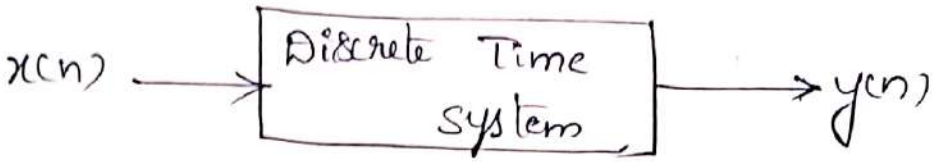
$$y(t) = T[x(t)]$$



Discrete Time System:-

A discrete time s/m is one which operates on a discrete time sig and produces a discrete time output sig. If the input and output of discrete time systems are $x(n)$ and $y(n)$, then we say that $x(n)$ is transformed to $y(n)$ i.e.,

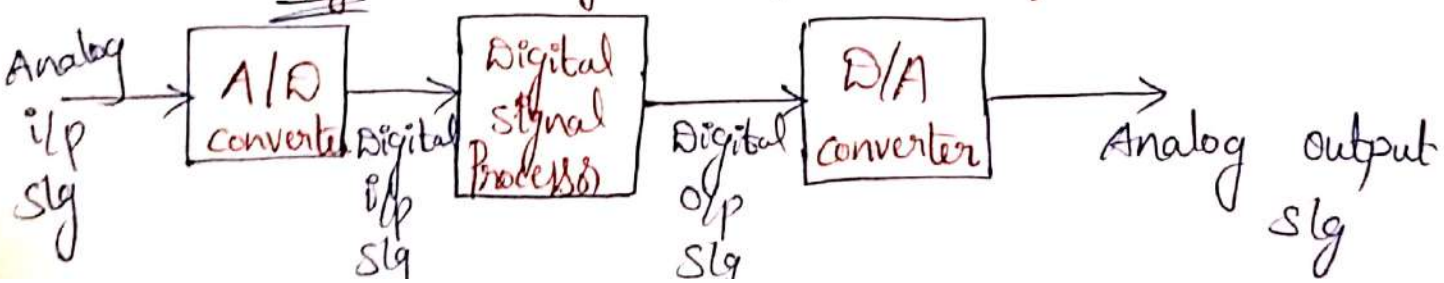
$$y(n) = T[x(n)]$$



Block Diagram of Digital Signal Processing System:

Signal processing is a method of extracting information from the sig which inturn depends on the type of sig and the nature of information it carries. Thus sig processing is connected with representing sig's in the mathematical terms and extracting the information by carrying out algorithmic operations on the sig with the help of digital s/m's like personal computer and systems designed using digital ICs, Microprocessors and Micro controllers.

fig: Block diagram of DSP system.



The basic components of a DSP Sm are shown in above fig. The DSP Sm involves conversion of analog to digital Sg then processing of the digital Sg by a digital Sm then the conversion of the proceed digital Sg back to its original form i.e, analog Sg.

The real world Sgs are analog and for processing by digital Sms the Sgs are converted to digital form.

For conversion of Sgs from analog to digital on ADC is employed. The various steps in ADC conversion process are sampling and quantization of analog Sgs and then converting the quantized sampled to suitable binary codes. The digital signals in the form of binary codes are fed to digital systems for processing and after processing it generates on output digital signal in the form of binary codes.

5.5.12. Graphical

Representation of Discrete-Time Signals:-

Discrete-Time signals are signals which are defined only at discrete instants of time. For those signals, the amplitude btw the two time instants is just not defined for discrete-time signal the independent variable is time n, and it is represented by x(n).

There are four ways of representing DTS's.

1. Graphical representation
2. Functional representation
3. Tabular representation
4. Sequence representation.

1. Graphical Representation:-

Consider a single x(n) with values

$x(-2) = -3$, $x(-1) = 2$, $x(0) = 0$, $x(1) = 3$, $x(2) = 1$, $x(3) = 2$.

This discrete time sig can be represented graphically as shown in below.

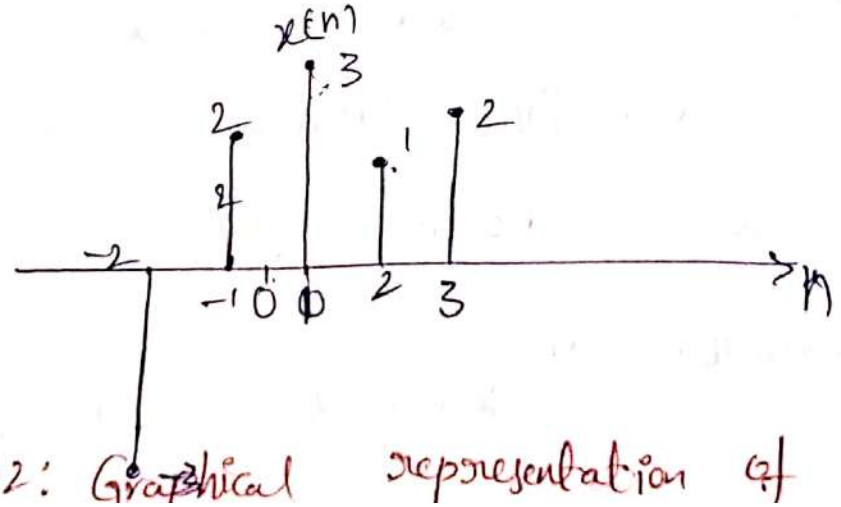


fig: 1.2: Graphical representation of DTS.

2. Functional Representation:-

The sig shown in fig 1.2 can be represented using the functional representation as follows:

$$x(n) = \begin{cases} -3 & \text{for } n = -2 \\ 2 & \text{for } n = -1 \\ 0 & \text{for } n = 0 \\ 3 & \text{for } n = 1 \\ 1 & \text{for } n = 2 \\ 2 & \text{for } n = 3 \end{cases}$$

3. Tabular Representation:- The sig shown in fig: 1:2 can be represented in Tabular form as follows.

n	-2	-1	0	1	2	3
x(n)	-3	2	0	3	1	2

4. Sequence Representation:- The sequence of fig 1.2 can be represented as.

$$x(n) = \{-3, 2, 0, \underset{\uparrow}{3}, 1, 2\}$$

The arrow mark t denotes the $n=0$ term, when no arrow is indicated, the first term corresponds to $n=0$. So a finite duration sequence, that specifies the condition $x(n)=0$ for $n<0$ can be represented as

$$x(n) = \{3, 5, 2, 1, 4, 7\}$$

Elementary Discrete time signals:

The standard DT Signals are as follows:

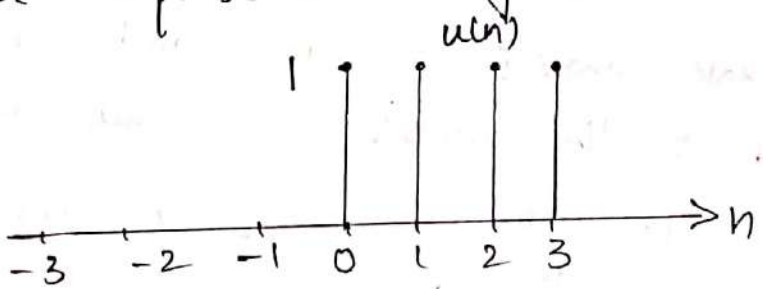
1. Unit step sequence
2. Unit ramp sequence
3. Unit parabolic sequence
4. Unit impulse sequence
5. Sinusoidal sequence
6. Real exponential sequence
7. Complex exponential sequence

1. Unit Step Sequence:-

The discrete-time unit step sequence $u(n)$ is defined as

$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

Graphical representation of $u(n)$ is shown in below



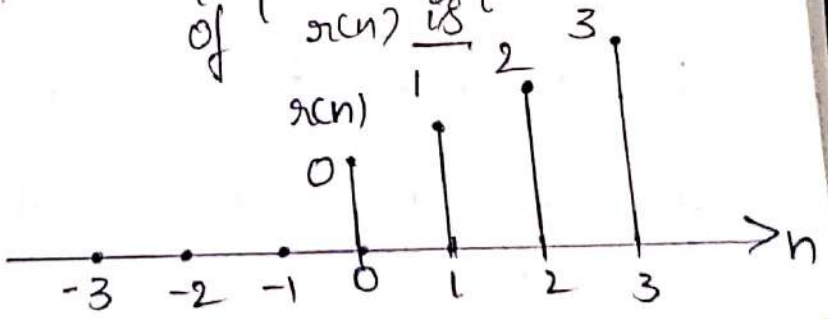
2. Unit Ramp Sequence:-

A discrete unit ramp function is denoted as

$r(n)$ can be defined as.

$$r(n) = \begin{cases} n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

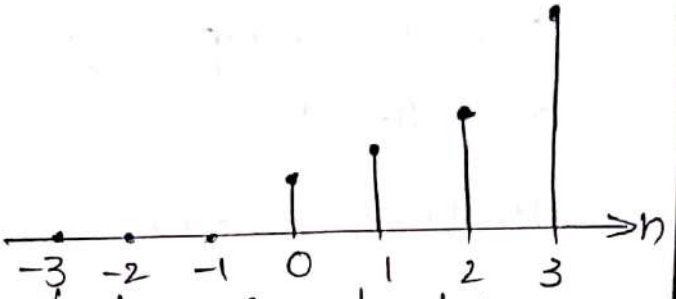
Graphical representation of $r(n)$ is



3. Unit parabolic sequence:-

A Discrete time unit parabolic function is denoted as $p(n)$ and can be defined as

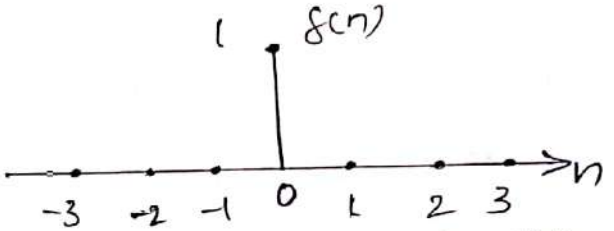
$$p(n) = \begin{cases} \frac{n^2}{2} & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$



4. Unit Impulse Sequence:-

A Discrete unit Impulse function is denoted as $\delta(n)$ and can be defined as

$$\delta(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



5. Sinusoidal sequence:-

The discrete-time sinusoidal sequence is given by:

$$x(n) = A \sin(\omega n + \phi)$$

where, A is the amplitude, ω is angular frequency, ϕ is phase angle in radians and n is an integer.

The period of the discrete-time sinusoidal sequence is

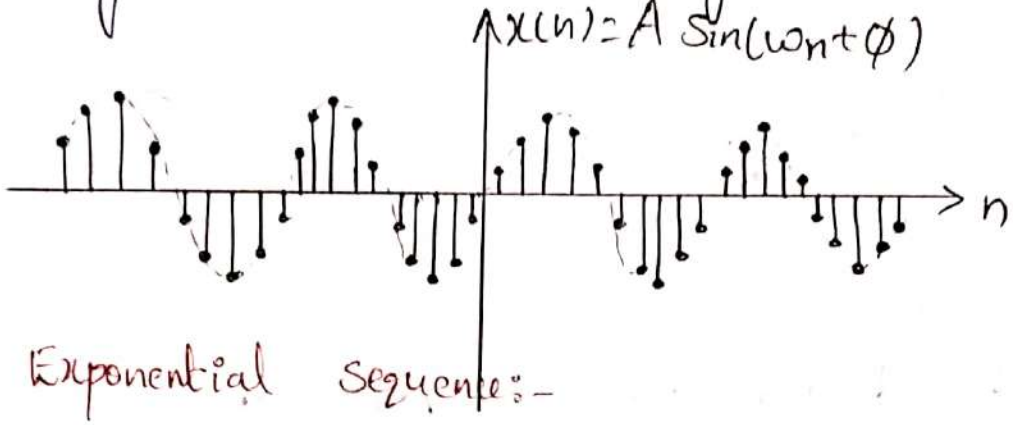
$$N = \frac{2\pi m}{\omega}$$

, where N and m are integers

All continous-time sinusoidal signals are periodic, but discrete-time sinusoidal sequences may (or) may not be periodic depending on the value of 2π.

Eg. a discrete time signal to be periodic, the angular frequency must be a rational multiple of 2π.

The graphical representation of a discrete-time sinusoidal signal is shown in figure.

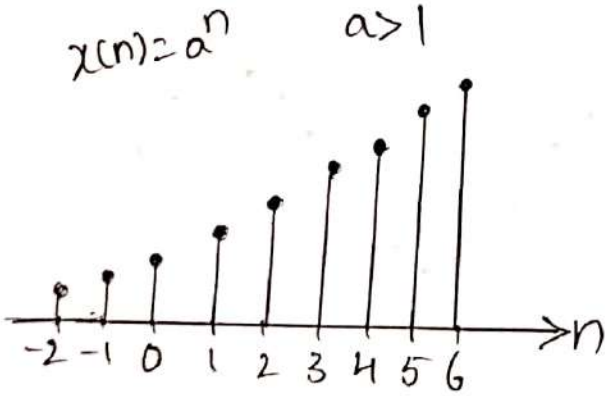


6- Real Exponential Sequence:-

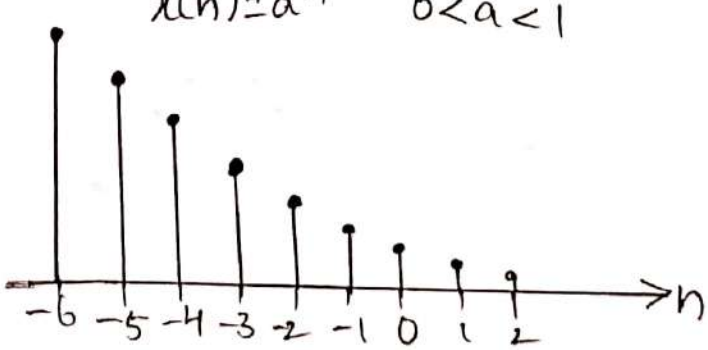
The discrete-time real exponential sequence is defined as $x(n) = a^n$ for all n.

The below figures illustrates different types of discrete time exponential signals.

* when $a>1$, the sequence grown exponentially as shown in fig (a).
* when $0<a<1$, the sequence delays exponentially as shown in figure (b).
* when $a<0$, the sequence takes alternating signs as shown in fig (c) and (d).

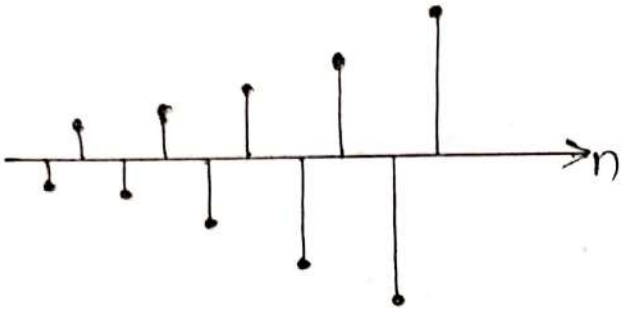


(a)



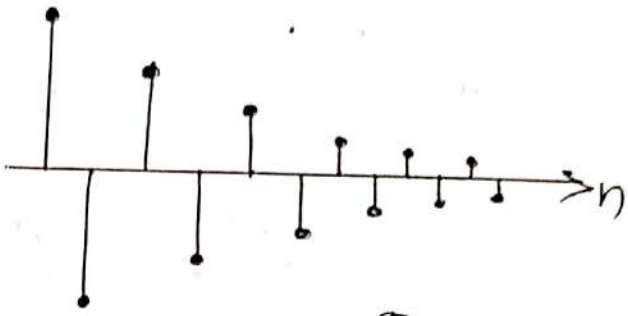
(b)

$x(n) = a^n \quad a < -1$



(c)

$x(n) = a^n \quad -1 < a < 0$



(d)

(7) Complex Exponential Sequence:-

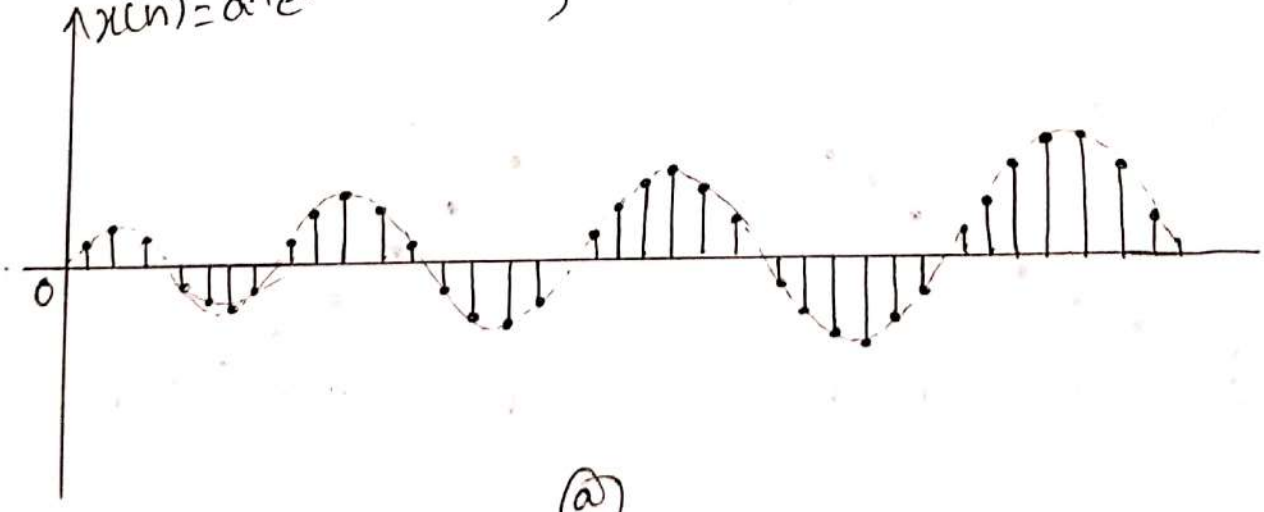
The discrete time complex exponential sequence is defined as

$$x(n) = a^n e^{j(\omega_0 n + \phi)}$$

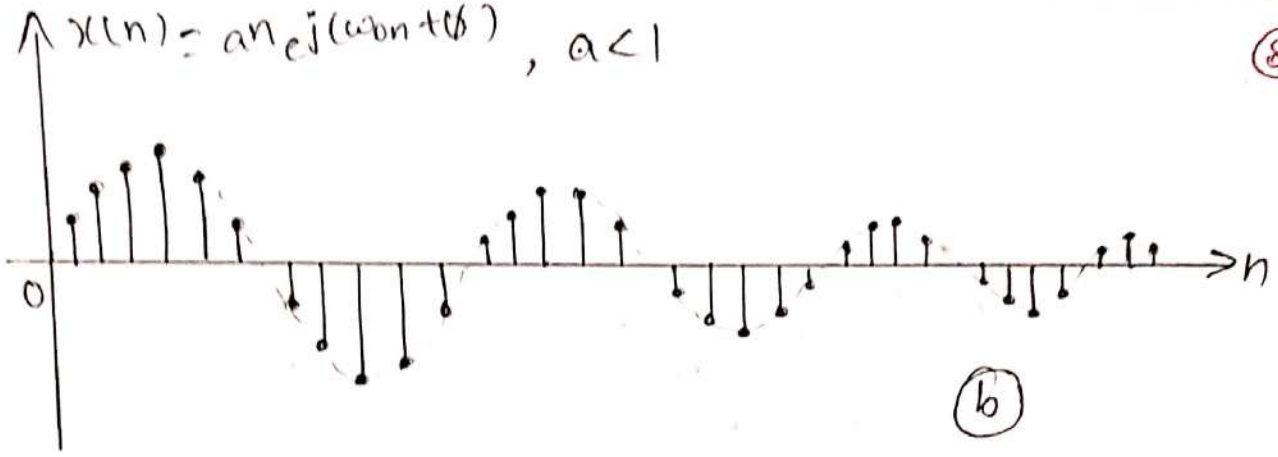
$$= a^n \cos(\omega_0 n + \phi) + j\, a^n \sin(\omega_0 n + \phi)$$

* For $|a| = 1$, the real and imaginary parts of complex exponential sequence are sinusoidal.
* For $|a| > 1$, the amplitude of the sinusoidal sequence exponentially grows as shown in fig (a).
* For $|a| < 1$, the amplitude of the sinusoidal sequence exponentially delays as shown in fig (b)

$x(n) = a^n e^{j(\omega_0 n + \phi)}, \quad a > 1$



(a)



(b)

Classification of Discrete Time Signals:-

Based on their characteristics the discrete time signals are classified into the following types.

1. Deterministic & Non-deterministic/Random signal.
2. Energy & Power signals.
3. Symmetric [Even] & Assymmetric [odd] signals.
4. Periodic & Aperiodic signals.
5. Causal & Non-causal signals.

Deterministic & Non-Deterministic (or) Random signals-

Deterministic signal:-

The signal that can be completely represented by mathematical equations are called deterministic signal.

Eg:- Step, Step, ramp, parabolic, Sinusoidal, impulse etc.

Non-Deterministic (or) Random signal:-

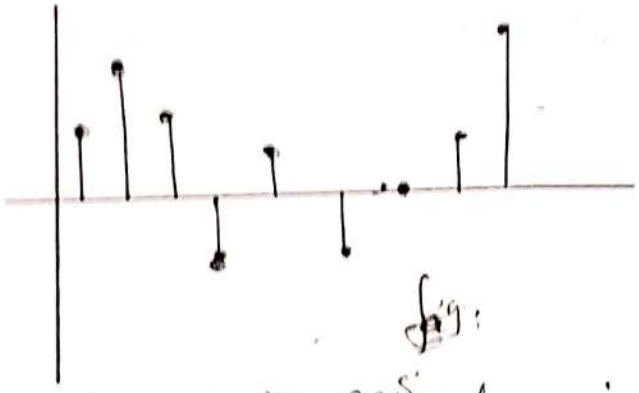
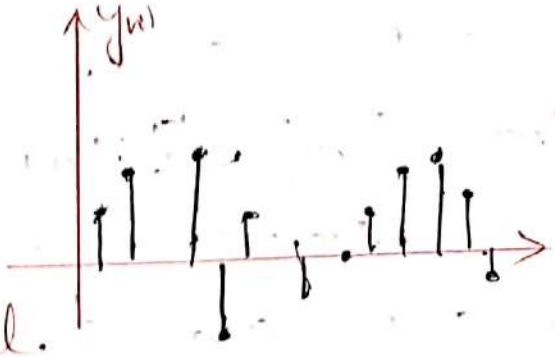


fig:

fig: Noise signal.

The signal that cannot be represented by the mathematical eq's are called non-deterministic (or) random sig.



(2) Energy & Power signal.

Energy signal:-

If the energy of a signal is finite, then the signal is said to be an energy signal.

For energy signal, power, $P=0$.

Energy of a signal is

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 \rightarrow \text{Eq (1)}$$

Power Signal:-

If the power of a signal is finite, then the signal is said to be a power signal.

For power signal; Energy, $E = \infty$

∴ Power of a signal is

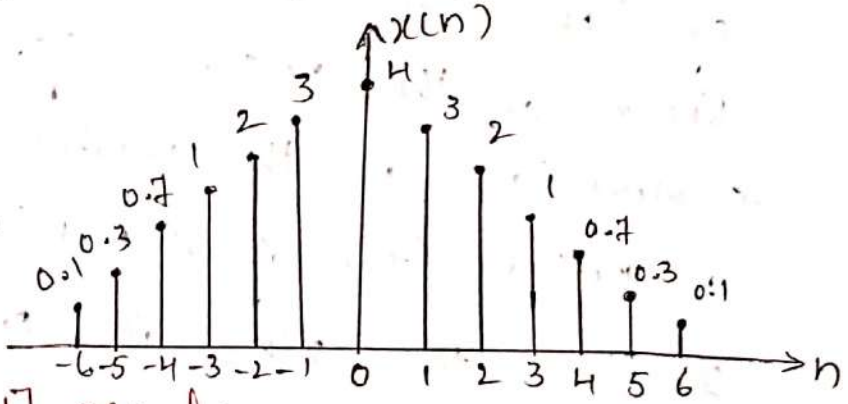
$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 \quad \rightarrow (2)$$

3. Symmetric [even] & Asymmetric [odd] signals:

Symmetric [Even] signal:-

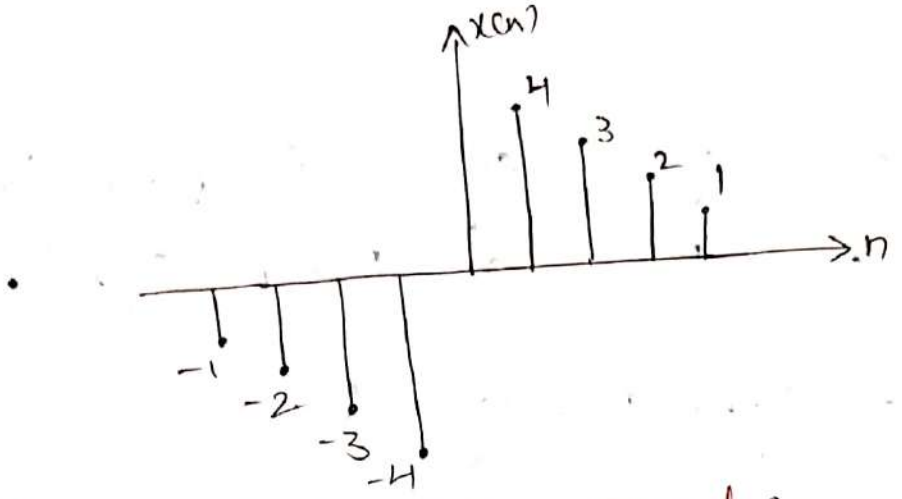
A signal $x(n)$ is said to be a symmetric sig, when it specifies the condition $x(-n) = x(n)$ for all values of "n".

Eg: $\cos \omega_0 n$.



Asymmetric [odd] signal:-

A signal $x(n)$ is said to be an asymetric [odd] signal, when it satisfies the condition $x(-n) = -x(n)$ for all values of n.



4. Periodic & Aperiodic Signals:-

Periodic signal:-

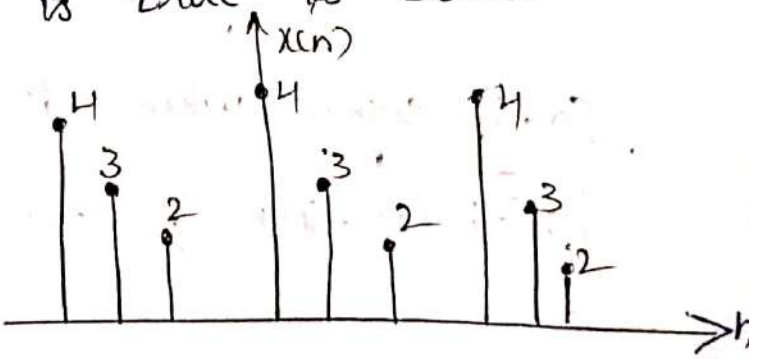
A signal which has a definite pattern and repeats itself at regular intervals of time is called a periodic signal.

(or)

when a discrete time signal $x(n)$ satisfies the condition $x(n+N) = x(n)$ for integer value of "N", then the discrete time signal is known as the periodic sig.

* Here, "N" is the number of samples of a period i.e., if $x(n+N) = x(n)$ for all values of "n" then $x(n)$ is periodic.

* The smallest value of "N" for which the above equation is true is called fundamental period.

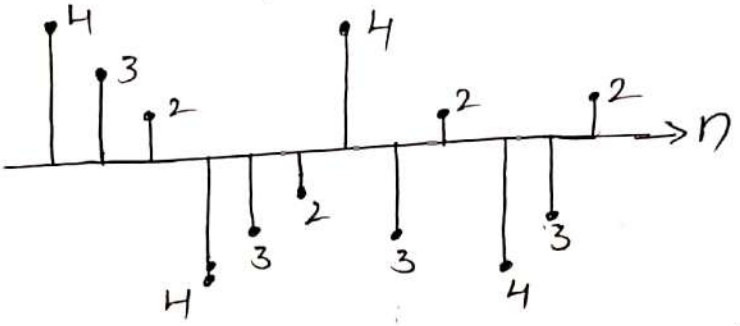


Aperiodic Signal:-

A signal which does not repeat at regular intervals of time is called an aperiodic signal.

(or)

If there is no value of "N" that satisfies the above equation, then x(n) is called an aperiodic signal.



Note:

* The periodic signals are power signals
* The discrete time sinusoidal signal and complex exponential signals are periodic signals when x_0 is a rational number.

5. Causal & Non-causal signals:-

Causal signal:-

A signal x(n) is said to be a causal, if its value is zero (or) for $n<0$.

Eg:- $x(n) = \{2, 4, 1, 4, 3\}$
$\uparrow$

$x(n) = \{2, 3, 4, 1, 2\}$
$\uparrow$

causal

Non - causal Signal :-

A signal $x(n)$ is said to be a non-causal, if its value is zero (0) for $n \geq 0$.

Eg: $x(n) = \{-1, -2, 1, 2, 4\}$

$x(n) = \{4, 3, 2, 1, 3, 2\}$

Non - causal

(1) Determine the values of power and energy of the following signal.

(a) $x(n) = \left(\frac{1}{3}\right)^n . u(n)$.

Given. $x(n) = \left(\frac{1}{3}\right)^n u(n)$

$$\text{By } E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=-\infty}^{\infty} \left|\left(\frac{1}{3}\right)^n . u(n)\right|^2$$

$$= \sum_{n=-\infty}^{0} \left|\left(\frac{1}{3}\right)^n . u(n)\right|^2 + \sum_{n=0}^{\infty} \left|\left(\frac{1}{3}\right)^n u(n)\right|^2$$

$$E = 0 + \sum_{n=0}^{\infty} \left|\left(\frac{1}{3}\right)^n . u(n)\right|^2$$

$$= \sum_{n=0}^{\infty} \left|\left(\frac{1}{3}\right)^n u(n)\right|^2$$

$$= \sum_{n=0}^{\infty} \left|\left(\frac{1}{3}\right)^1 . 1\right|^2$$

$$= \sum_{n=0}^{\infty} \left|\left(\frac{1}{3}\right)\right|^2$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{9}\right)^2$$

$$E = \left(\frac{1}{9}\right)^0 + \left(\frac{1}{9}\right)^1 + \left(\frac{1}{9}\right)^2 + \cdots + \infty$$

$$= 1 + \frac{1}{9} + \left(\frac{1}{9}\right)^2 + \cdots + \infty$$

$$= \frac{1}{1-\left(\frac{1}{9}\right)} \qquad \left[\because 1 + a + a^2 + a^3 + \cdots + \infty = \frac{1}{1-a}\right]$$

$$= \frac{1}{8/9}$$

$$E = 9/8 \text{ (finite)}$$

WKT power

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left|\left(\frac{1}{3}\right)^n . u(n)\right|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} \left|\left(\frac{1}{3}\right)^n\right|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} \left|\left(\frac{1}{9}\right)^n\right|$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \left[\frac{1-\left(\frac{1}{9}\right)^{n+1}}{1-1/9}\right]$$

$$\boxed{P = 0} \qquad \left(\because \sum_{n=0}^{N} a^n = \frac{1-a^{n+1}}{1-a}\right)$$

As E = finite, P = 0 then it is energy signal.

(2) $x(n) = e^{2n}\, u(n)$

Energy $E = \sum_{n=-\infty}^{\infty} |x(n)|^2$

$$= \sum_{n=-\infty}^{\infty} |e^{2n}\, u(n)|^2$$

$$= \sum_{n=0}^{\infty} |e^{2n}\, u(n)|^2$$

$$= \sum_{n=0}^{\infty} |e^{2n}|^2$$

$$= \sum_{n=0}^{\infty} |e^{4n}|$$

$$= 1 + e^4 + e^8 + \cdots \infty$$

$E = \infty$ (Infinity)

WKT, power $P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} |e^{2n}\, u(n)|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} |e^{2n}(1)|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{\infty} |e^{4n}|$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \left[\frac{e^{4(N+1-1)}}{e^4 - 1}\right]$$

$$\boxed{P = \infty}$$

∴ The signal is neither energy (nor) power signal.

(3) Determine whether the following signal is periodic (or) not. If a signal is periodic find its fundamental period.

(a) $x(n) = e^{j6\pi n}$

Given

$x(n) = e^{j6\pi n}$

This is the form of $e^{j\omega n}$.

then, $\omega_0 = 6\pi$

$\therefore$ The ω_0 is integral multiple of π.

So, the given signal is periodic

WKT, $N = 2\pi \left[\frac{m}{\omega_0}\right]$

$N = 2\pi \left[\frac{m}{6\pi}\right]$

$N = \frac{m}{3}$

if $m = 3 \Rightarrow N = 1$

Then fundamental period is $\boxed{N=1}$

(2) $x(n) = e^{j\,3/5\,(n+1/2)}$

Given $x(n) = e^{j\,3/5\,(n+1/2)}$

$\omega_0 = 3/5$

ω_0 is not an integral multiple of π.

So, the signal is aperiodic.

Operations on signals:-

Signal processing is a group of basic operations applied to an input signal resulting in another signal as an output. The mathematical transformation from one signal to another is represented as

$$\boxed{y(n) = T[x(n)]}$$

The basic set of operations are

1. Time Shifting
2. Time Reversal
3. Time scaling
4. Scalar multiplication
5. Signal multiplier
6. Signal Addition.

(1) Time Shifting:-

The shift operation takes the input sequence and shifts values by an interger increment "K" of the dependant variable.

* The shifting may delay (&) advance the sequences in time. Mathematically it can be written as

$y(n) = x(n-k)$

where, $x(n)$ is an input signal.

$y(n)$ is an output signal.

* If 'K' is positive, the shifts delays the sequence
* If 'K' is negative, the shifts advances the sequence.
* A Signal $x(n)$ is shown in figure.
* The signal $x(n-3)$ is obtained by shifting $x(n)$ right by 3 units of time.
* The signal $x(n+2)$ is obtained by shifting $x(n)$ left by 2 units of time.

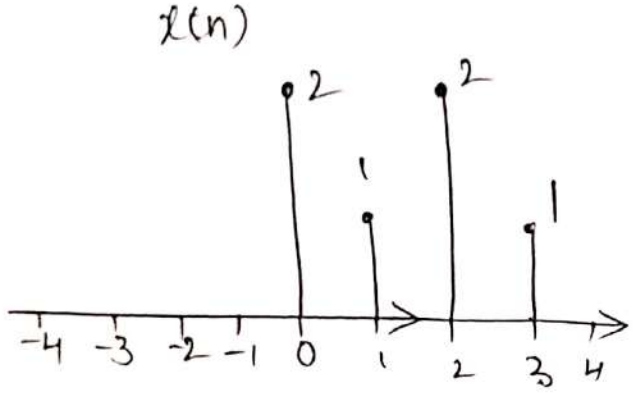


Fig: Original x(n)

$x(n-3) \rightarrow x(n-k) \Rightarrow k=3$

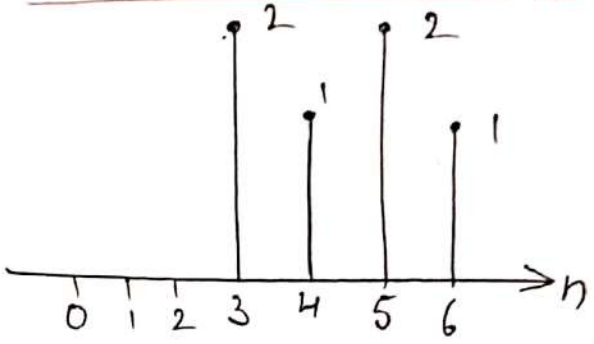


Fig: Delayed version of x(n)

$x(n+2) \rightarrow x(n-k) \Rightarrow k=-2$

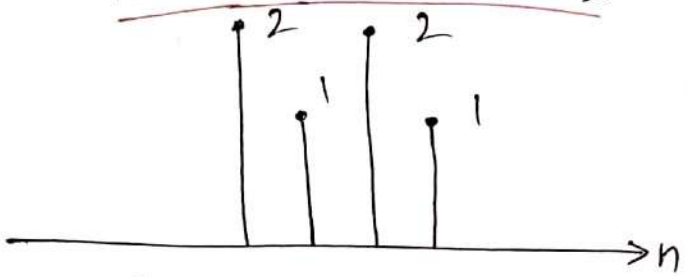
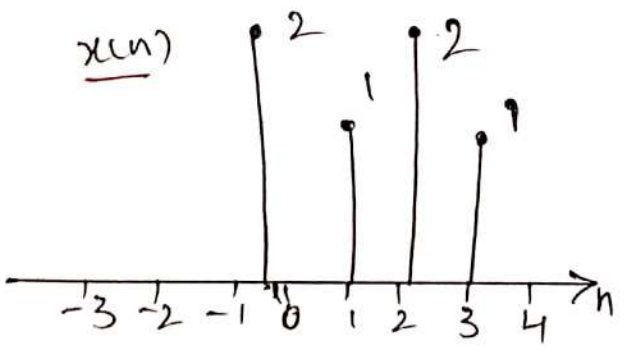
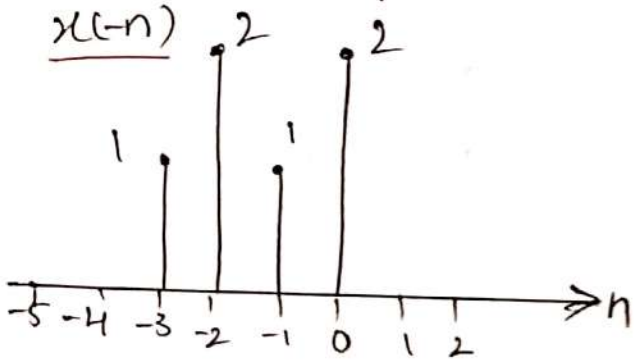


Fig: Advanced version of x(n)

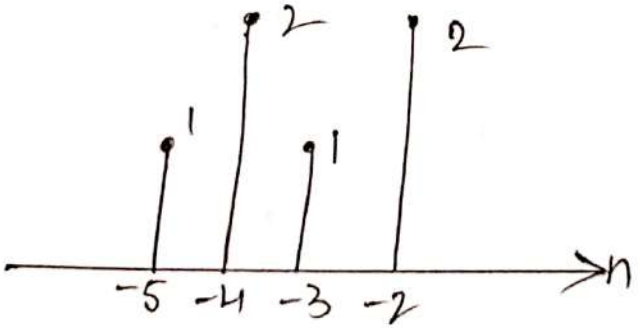
2. Time Reversal (Folding):-

* The time reversal of sequence x(n) can be obtained by folding the sequence about n=0.
* It is denoted by x(-n).
* For given x(n), x(-n) is shown in figure.



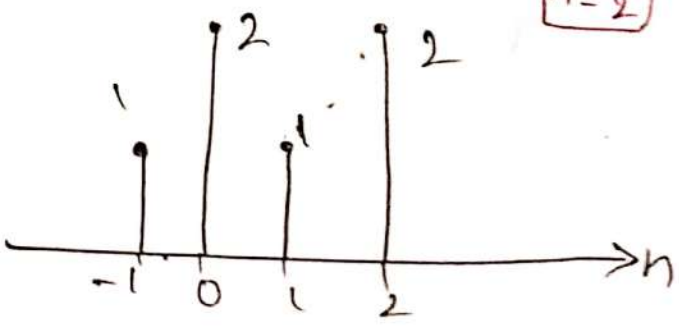


$x(-n-2)$; $-n-2=0 \Rightarrow -n=2 \Rightarrow \boxed{n=-2}$



$x(-n+2)$; $-n+2=0 \Rightarrow +n=-2$

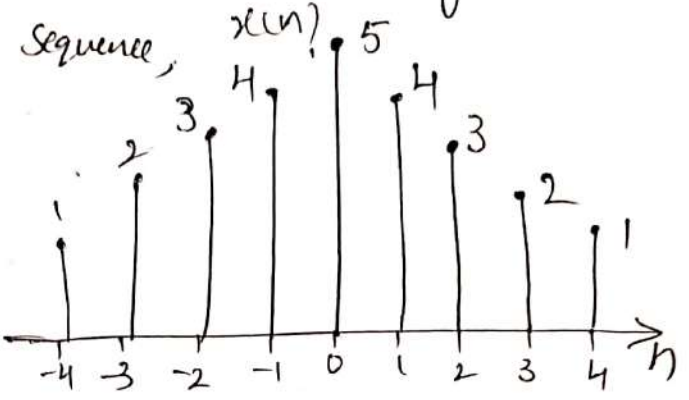
$= \boxed{n=2}$



3. Time Scaling:-

* It is obtained by replacing "n" by "λn" in the sequence x(n).

* Let x(n) is a sequence shown in figure.



If λ=2, we get a new sequence,

$y(n) = x(2n)$

$y(n) = x(\lambda n)$

$\lambda = 2$

$y(n) = x(2n)$

$n = -4 ; y(-4) = x(-8) = 0$

$n = -3 ; y(-3) = x(-6) = 0$

$n = -2 ; y(-2) = x(-4) = 1$

$n = -1 ; y(-1) = x(-2) = 3$

$n = 0 ; y(0) = x(0) = 5$

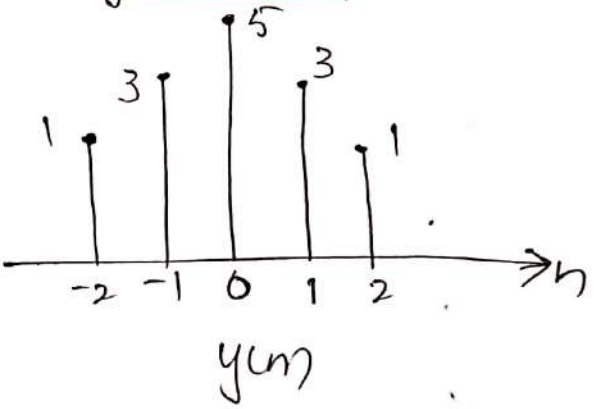
$n = 1 ; y(1) = x(2) = 3$

$n = 2 ; y(2) = x(4) = 1$

$n = 3 ; y(3) = x(6) = 0$

$n = 4 ; y(4) = x(8) = 0$

$y(n) = x(2n)$



y(n)

4. Scalar Multiplication:-

In scalar multiplication x(n) is multiplied by a scalar value "a" to get y(n).

$$x(n) \xrightarrow{a} y(n) = a\,x(n)$$

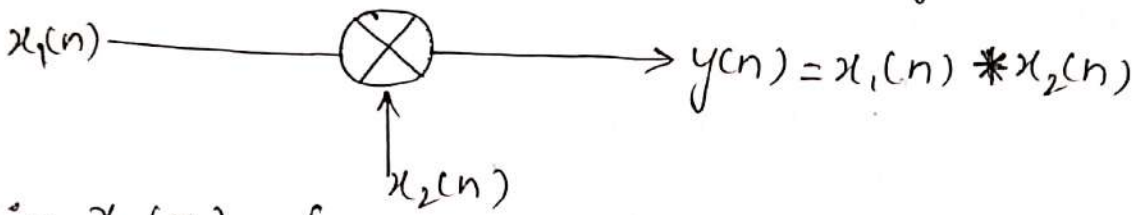
Ex: Let $x(n) = \{-1, -2, -3, \underline{4}, 2\}$

$a = 2$

$y(n) = a\,x(n) = \{-2, -4, -6, 2, 4\}$

5. Signal Multiplier:-

In this two signals $x_1(n)$ & $x_2(n)$ are multiplied to get a new sequence $y(n)$.



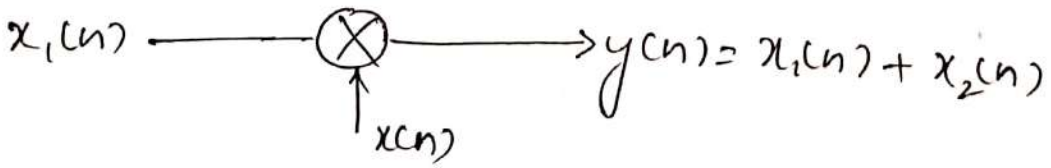
Ex:- $x_1(n) = \{-4, -3, -2, 1\}$

$x_2(n) = \{-1, -1, -1, -1\}$

$y(n) = x_1(n) * x_2(n) = \{4, 3, 2, -1\}$

6. Signal Addition:-

In this two signals $x_1(n)$ & $x_2(n)$ are addition to get a new sequence $y(n)$.



Ex:- $x_1(n) = \{-1, -2, 1, 2\}$

$x_2(n) = \{1, 1, 1, 1\}$

$y(n) = x_1(n) + x_2(n) = \{0, -1, 2, 3\}$

Discrete Time Systems :-

A discrete time system is a device algorithm that operates on discrete time i/p seq $x(n)$ according to well defined rules to produce another discrete time signals $y(n)$ as an output signal.

* The relationship between $x(n)$ & $y(n)$ is

$$\boxed{y(n) = T[x(n)]}$$

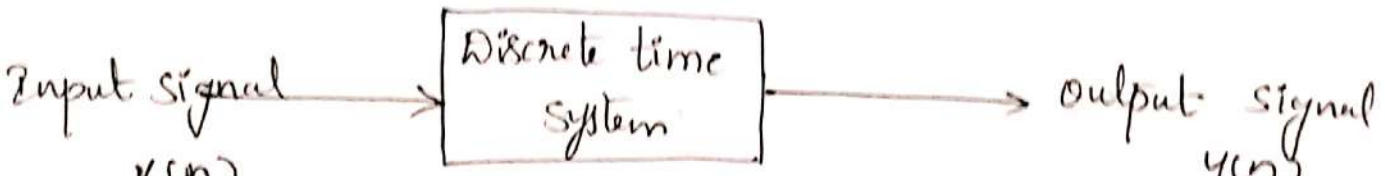


fig: Discrete time system

Classification of Discrete time systems :-

According to the general characteristics, the systems are classified as,

1. Static and dynamic systems.
2. Causal and Non-causal systems
3. Linear and Non-linear systems
4. Time variant and Time Invariant systems
5. FIR and IIR systems
6. Stable and Unstable systems.

1. Static & Dynamic systems :-

Static (Memory less) System:-

A discrete time system is called Static (or) memoryless, if its output at any instant "n" depends on the input at the same time but not on past (or) future samples of input.

Ex:- $y(n) = a \cdot x(n)$
$y(n) = a^2 \cdot x(n)$
} Memoryless (or) Static System

Dynamic (Memory) System:-

A discrete time system is called dynamic (or) memory, if its output at any instant "n" depends on the past & future inputs but not on present sample of input.

Ex: $y(n) = x(n-1) + x(n-2)$
$y(n) = x(n+1) + x(n)$
} memory (or) Dynamic System.

* Determine whether the systems are static (or) dynamic.

(1) $y(n) = x(n) \cdot x(n-1)$

Given $y(n) = x(n) \cdot x(n-1)$.

for $n=0$

$y(0) = x(0) \cdot x(0-1)$

$y(0) = x(0) \cdot x(-1)$

∴ For the present input of n, the output depends on present as well as past inputs so, the system is dynamic system.

(2) $y(n) = x^2(n) + x(n)$

Given

$y(n) = x^2(n) + x(n)$

for $n=0$

$y(0) = x^2(0) + x(0)$

∴ The output depends on present inputs only so, the system is static system.

2. Causal & Non-Causal Systems:-

Causal system:-

A system is said to be a causal, if the output of the system at any instant of time "n" depends only at present and past inputs but not an future inputs, which can be represented as $y(n) = F[x(n), x(n-1), x(n-2) \cdots]$

Non-Causal System:-

If the output of a system depends on future inputs then it is said to be a non-causal system.

* Determine whether the following systems are causal (or) Non-causal.

1. $y(n) = x(n) + \frac{1}{x(n-1)}$

Given $y(n) = x(n) + \frac{1}{x(n-1)}$

for $n = -1 \Rightarrow y(-1) = x(-1) + \frac{1}{x(-2)}$

$n = 0 \Rightarrow y(0) = x(0) + \frac{1}{x(-1)}$

$n = 1 \Rightarrow y(1) = x(1) + \frac{1}{x(0)}$

For all values of "n", the system depends on present and past inputs therefore, it is a causal sm.

2. $y(n) = x(n^2)$

Given $y(n) = x(n)^2$

For $n = -1 \Rightarrow y(-1) = x(-1)^2 = x(1)$

$n = 0 \Rightarrow y(0) = x(0)^2 = x(0)$

$n = 1 \Rightarrow y(1) = x(1)^2 = x(1)$

For all values of "n", the system depends on future inputs. So, the system is a non-causal sm.

3. FIR & IIR systems :-

FIR (Finite Impulse Response) system:-

If the impulse response of the system is of finite duration then that system is known as FIR system.

Ex: $h(n) = \begin{cases} 1, & n = -1, 2 \\ 2, & n = 1 \\ 3, & n = 0, 3 \\ 0, & \text{otherwise} \end{cases}$

IIR (Infinite Impulse Response) System:-

If the impulse response of the system is of infinite duration then that the system is known as IIR system.

Ex:- $h(n) = a^n . u(n)$.

4. Stable & Unstable System:-

An LTI system is stable, if it produces a bounded output sequence for every bounded input sequence. If for same bounded input sequence $x(n)$, the output is unbounded [infinity] then the system is said to be unstable.

* Let $x(n)$ be a bounded input sequence, $h(n)$ be the impulse response of a S/m, $y(n)$ be the output sequence.
* The necessary and sufficient condition for the stability is $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$.

* Test the stability of system whose impulse response is $h(n) = (\frac{1}{2})^n . u(n)$.

Given, $h(n) = (\frac{1}{2})^n . u(n)$.

To test the stability of a system

$$\sum_{n=-\infty}^{\infty} |h(n)|$$

$$h(n) = \sum_{n=-\infty}^{\infty} \left|(\tfrac{1}{2})^n . u(n)\right|$$

$$= \sum_{n=-\infty}^{0} \left|(\tfrac{1}{2})^n . u(n)\right| + \sum_{n=0}^{\infty} \left|(\tfrac{1}{2})^n . u(n)\right|$$

$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

$$= 0 + \sum_{n=0}^{\infty} \left|(\tfrac{1}{2})^n . u(n)\right|$$

$$= \sum_{n=0}^{\infty} \left|(\tfrac{1}{2})^n . 1\right|$$

$$= \sum_{n=0}^{\infty} (\tfrac{1}{2})^n .$$

$$= (\tfrac{1}{2})^0 + (\tfrac{1}{2})^1 + (\tfrac{1}{2})^2 + \text{----} + \infty$$

$$= 1 + (\tfrac{1}{2}) + (\tfrac{1}{2})^2 + \text{---} - + \infty$$

$$= \frac{1}{1-a} = \frac{1}{1-\frac{1}{2}} = \frac{2}{1} \Rightarrow \boxed{2 < \infty}$$

$\therefore$ This system is a stable.

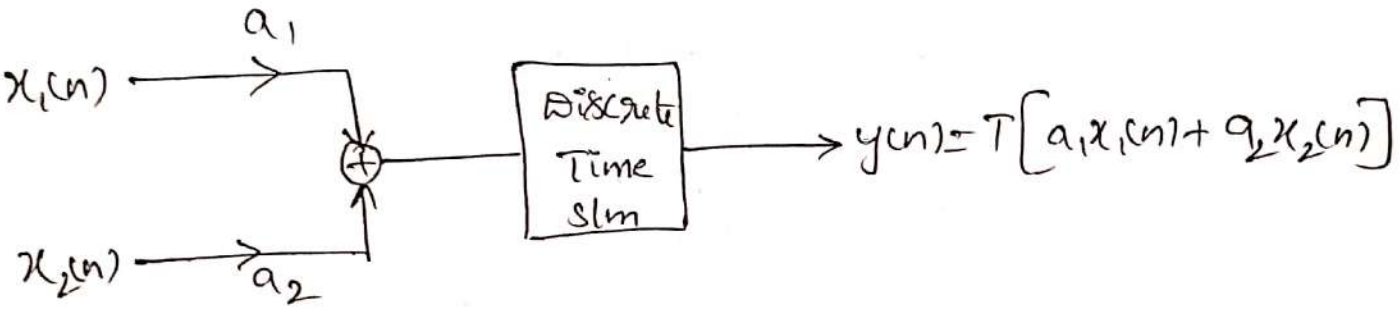
5. Linear & Non-Linear System:-

* A system is said to be a linear system if it satisfies the superposition theorem principle.
* Let $x_1(n)$ & $x_2(n)$ be the two input sequences then the system is said to be a linear, if and only if.

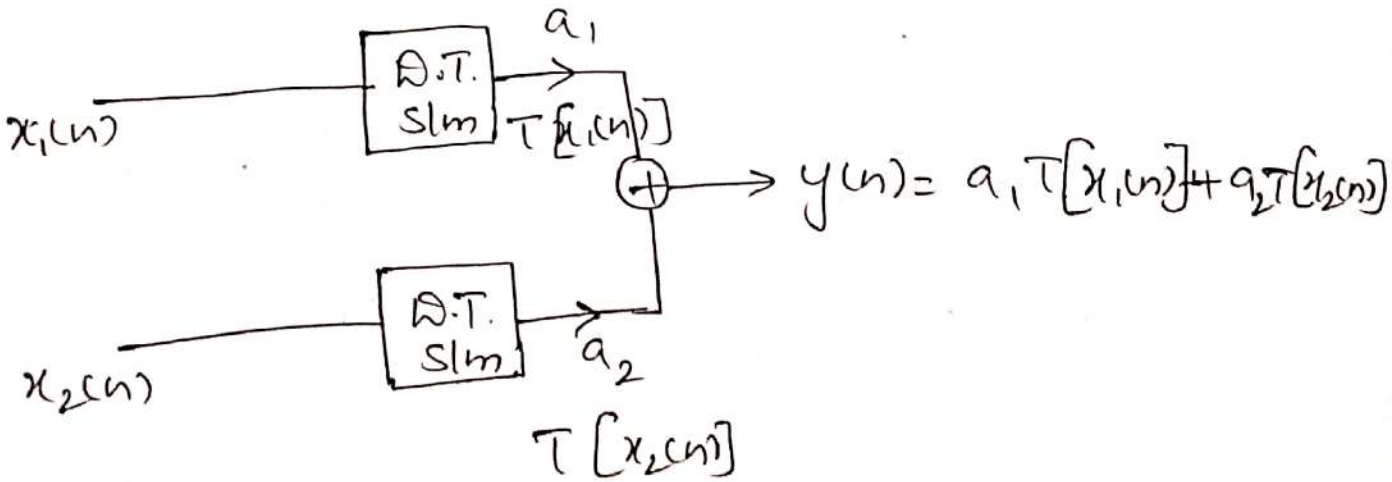
$$T[a_1 x_1(n) + a_2 x_2(n)] = a_1 T[x_1(n)] + a_2 T[x_2(n)]$$

* Here, a_1 & a_2 are arbitary constants.

Case (i):-



Case (ii):-



* Check the linearity of a given system.

① $y(n) = x(n) + \frac{1}{x(n-1)}$

Given $y(n) = x(n) + \frac{1}{x(n-1)}$

Let us consider $x_1(n)$ & $x_2(n)$ be the two i/p sequences.

$$y_1(n) = T[x_1(n)] = x_1(n) + \frac{1}{x_1(n-1)}$$

$$y_2(n) = T[x_2(n)] = x_2(n) + \frac{1}{x_2(n-1)}$$

Take LHS $= T[a_1 x_1(n) + a_2 x_2(n)]$

$$\Rightarrow T\left[a_1\left[x_1(n) + \frac{1}{x_1(n-1)}\right] + a_2\left[x_2(n) + \frac{1}{x_2(n-1)}\right]\right]$$

$$= a_1 x_1(n) + \frac{a_1}{a_1 x_1(n-1)} + a_2 x_2(n) + \frac{1}{a_2 x_2(n-1)}$$

Take RHS $= a_1 T[x_1(n)] + a_2 T[x_2(n)]$

$$\Rightarrow a_1\left[x_1(n) + \frac{1}{x_1(n-1)}\right] + a_2\left[x_2(n) + \frac{1}{x_2(n-1)}\right]$$

$$RHS = a_1 x_1(n) + \frac{a_1}{x_1(n-1)} + a_2 x_2(n) + \frac{a_2}{x_2(n-1)}$$

On comparing RHS & LHS.

LHS $\neq$ RHS.

∴ The system is not linear.

② $y(n) = x^2(n)$

Given $y(n) = x^2(n)$.

$y_1(n) = T[x_1(n)] = x_1^2(n)$

$y_2(n) = T[x_2(n)] = x_2^2(n)$

Take LHS $= T[a_1 x_1(n) + a_2 x_2(n)]$

$= T[a_1 [x_1^2(n)] + a_2 (x_2^2(n)]$

$= [a_1 x_1^2(n) + a_2 x_2^2(n)]$

Take RHS $= a_1 T[x_1(n)] + a_2 T[x_2(n)]$

$= a_1 [x_1^2(n)] + a_2 [x_2^2(n)]$

On comparing RHS & LHS

∴ The system is linear.

6. Time Variant & Time Invariant system :

* A slm is said to be a time invariant, if the characteristics of the system do not change with time i.e., $x(n) \xrightarrow{T} y(n)$.
* Then the system is invariant if and only if $x(n-k)$ will gives the $y(n-k)$.

$$x(n-k) \xrightarrow{T} y(n-k)$$

* This means if the input is shifted by "k" samples then the o/p is also shifted by the same number of samples for time invariant system.

To Test for Time Invariant property:-

* Delay the input sequence by "k" samples and calculate the output as,

$$y(n,k) = T[x(n-k)].$$

* Delay output by "k" samples it will gives $y(n-k)$.
* Then the system is shift invariant (or) Time invariant. If $y(n,k) = y(n-k)$ for all the values of "n".
* And the system is time variant if $y(n,k)$ is not equal to $y(n-k)$.

$y(n,k) \neq y(n-k)$ even for single value of "k".

* Determine whether the following systems are time variant (or) time Invariant.

(1) $y(n) = x(n) + x(n-1)$

Given $y(n) = x(n) + x(n-1)$

$y(n) = T[x(n)]$

$y(n) = x(n) + x(n-1)$

If the input is delayed by "k" samples.

(i.e) $y(n,k) = x(n-k) + x(n-k-1) \longrightarrow (1)$.

If the output is delayed by "k" samples.

(ii) $y(n-k) = x(n) + x(n-1)$

$y(n-k) = x(n-k) + x(n-k-1) \longrightarrow (2)$

By eqn (1) & eqn (2) we get $y(n,k) = y(n-k)$

∴ The given sys is time invariant.

(2) $y(n) = x(-n)$.

Given $y(n) = x(-n)$

$$y(n) = T[x(n)]$$

If the input is delayed by "k" samples.

$$y(n,k) = x(n-k) \longrightarrow (1)$$

If the output is delayed by k samples.

$$y(n-k) = x(-n-k) \longrightarrow (2)$$

By eqn (1) & eqn (2), we get $y(n,k) = y(n-k)$.

∴ The system is Time-Invariant.

(3) $y(n) = x(n/2)$

Given $y(n) = x(n/2)$.

$$y(n) = T[x(n)] = x(n/2).$$

If the input is delayed by k samples.

$$y(n,k) = x\left(\frac{n}{2} - k\right) \longrightarrow (1)$$

If the output is delayed by k samples

$$y(n-k) = x\left(\frac{n-k}{2}\right) \longrightarrow (2)$$

From eqn (1) & eqn (2), $y(n,k) \neq y(n-k)$

∴ The system is Time variant.

(4) $y(n) = n \cdot x^2(n)$.

Given $y(n) = n \cdot x^2(n)$

$$y(n) = T[x(n)] = n \cdot x^2(n).$$

If the input is delayed by "k" samples

$$y(n,k) = x(n-k) \cdot x^2(n-k) \longrightarrow (1)$$

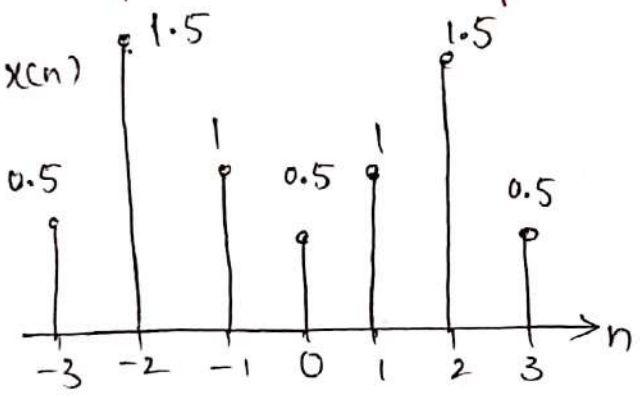
If the output is delayed by "k" samples.

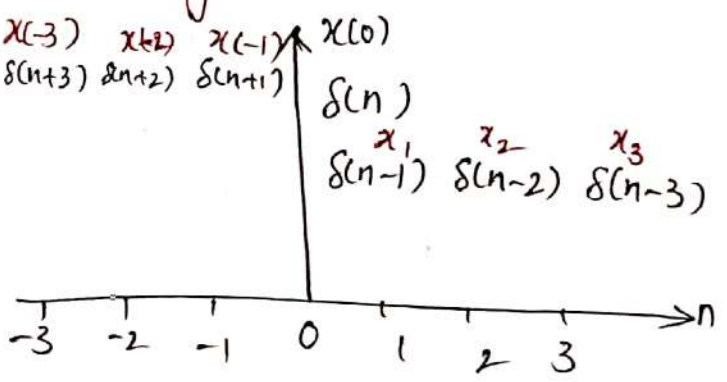
$$y(n-k) = (n-k) \cdot x^2(n-k) \longrightarrow (2)$$

From eqn (1) & eqn (2), $y(n,k) = y(n-k)$

∴ The system is time Invariant.

Representation of An Arbitary Sequence:-





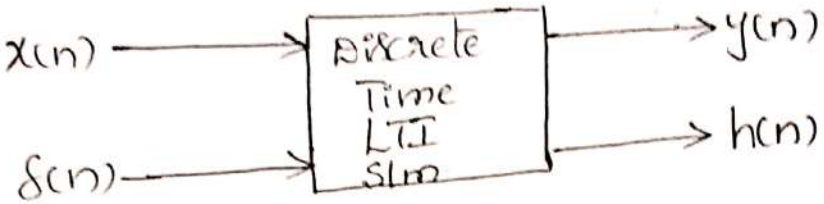
$$x(n) = x(-3)\cdot\delta(n-3) + x(-2)\cdot\delta(n+2) + x(-1)\cdot\delta(n+1) + x(0)\cdot\delta(0) + x(1)\cdot\delta(n-1) + x(2)\cdot\delta(n-2) + x(3)\cdot\delta(n-3) \quad \text{for} \quad -3 < n < 3$$

$$x(n) = \ldots\ldots x(-2)\cdot\delta(n+2) + x(-1)\cdot\delta(n+1) + x(0)\cdot\delta(0) + x(1)\cdot\delta(n+1) + x(2)\cdot\delta(n+2) \ldots \quad \text{for} \quad -\infty \ll 0 < \infty.$$

$$\boxed{x(n) = \sum_{k=-\infty}^{\infty} x(k)\cdot\delta(n-k)}$$

From the above equation we can say that any arbitary sequence x(n). can be represented as weighted sum of the impulses.

Analysis of Discrete Time LTI system:-



$$y(n) = T[x(n)] \longrightarrow (1)$$

T is a transfer operator.

$$h(n) = T[\delta(n)] \longrightarrow (2)$$

WKT, $x(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot \delta(n-k) \longrightarrow (3)$

from eqn (1) & (3) we get,

$$y(n) = T\left[\sum_{k=-\infty}^{\infty} x(k) \cdot \delta(n-k)\right]$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot T[\delta(n-k)] \longrightarrow (4)$$

$$h(n,k) = T[\delta(n-k)] \longrightarrow (5)$$

$$h(n-k) = h(n,k) \longrightarrow (6)$$

from eqn (5) & (6)

$$h(n-k) = T[\delta(n-k)] \longrightarrow (7)$$

from eqn (4) & (7)

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k) \longrightarrow (8)$$

$$y(n) = x(n) \cdot h(n)$$

$$y(n) = x(n) * h(n).$$

where "$*$" is convolution operator.

Convolution:-

Convolution is a mathematical way used to combine two different sequences $x(n)$ & $h(n)$ to produce a new sequence $y(n)$.

$$\therefore \boxed{y(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k)}$$

(or)

$$\boxed{y(n) = x(n) * h(n)}$$

Convolution

- Linear convolution
  1. Graphical method
  2. Tabular method
  3. Matrix method
- circular convolution
  1. concentric circle method
  2. Matrix method.

Linear convolution:-

Procedure for the calculation of convolution sum of two sequences.

Step 1:

Choose an initial value of n, the starting time for evaluating the output sequence $y(n)$. If $x(n)$ starts at $n=n_1$ and $h(n)$ starts at $n=n_2$ then $n=n_1+n_2$.

step 2:-

Express both sequences $x(n)$ & $h(n)$ interms of ader k.

step 3:-

Fold $h(k)$ about $k=0$ to obtain $h(-k)$ and shift by "n", to the right if n is positive and to the left if n is negative to obtain $h(n-k)$.

step 4:-

Multiply the two sequences $x(k)$ and $h(n-k)$ element by element and sum up the products to get $y(n)$.

step 5:-

Increment the index "n" and shift the sequence $h(n-k)$ to the right by one sample and do step 04.

Step 6:-

Repeat step-5 until the sum of products zero for all the remaining values of "n".

* The impulse response of an LTI slm is $h(n) = \{1, 3, 1, 3\}$, determine the response of the system to the input signal., $x(n) = \{3, 4, 2, 3\}$

A. Graphical Method:-

Step 1:- Given $h(n) = \{1, 3, +1, 3\} \Rightarrow n_1 = -2$

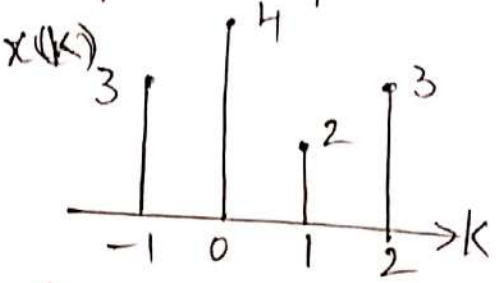
$x(n) = \{3, 4, 2, 3\} \Rightarrow n_2 = -1$

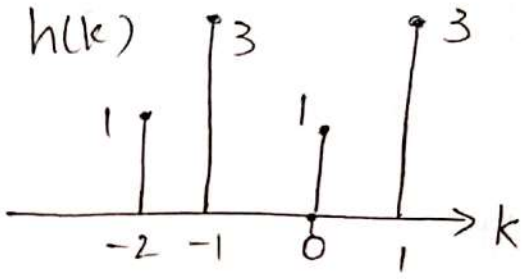
$n = n_1 + n_2$

$= -2 - 1$

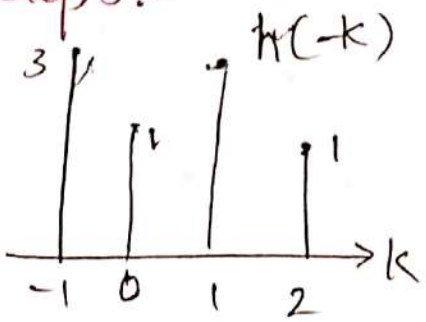
$\boxed{n = -3}$

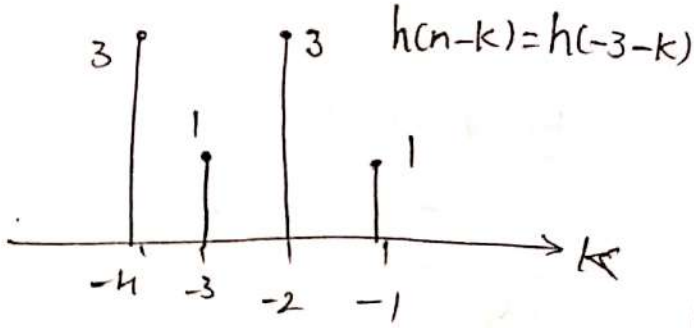
Step 2:- Express the $h(n)$ & $x(n)$ in terms of k.





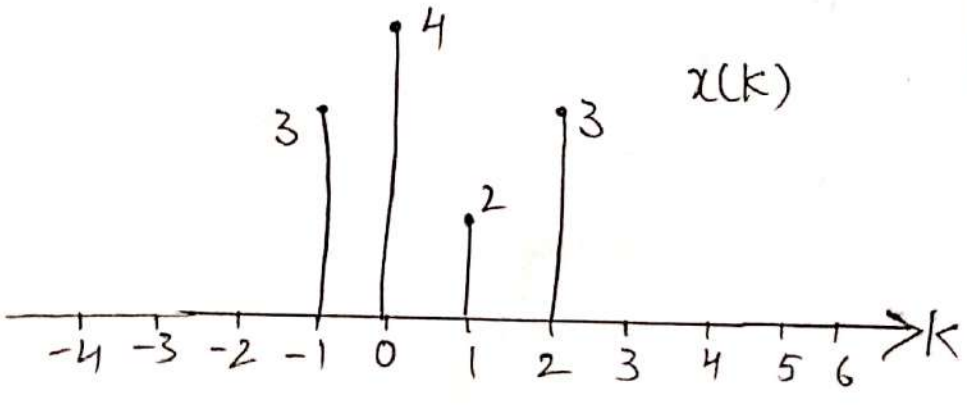
Step 3:-





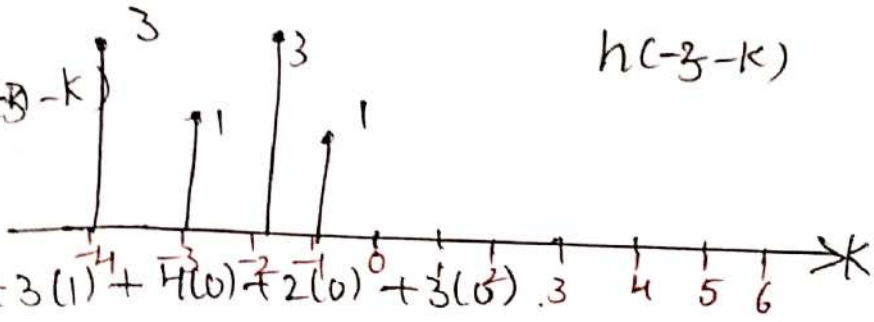
Step 4:-

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k)$$



At n = -3 :-

$$y(-3) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(-3-k)$$

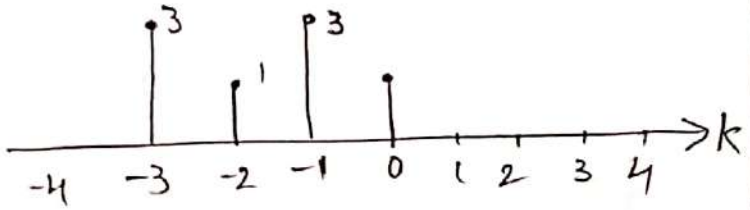


$$= 0(3) + 0(1) + 0(3) + 3(1) + 4(0) + 2(0) + 3(0)$$

$$\boxed{y(-3) = 3}$$

At n = -2 :-

$$y(-2) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(-2-k)$$

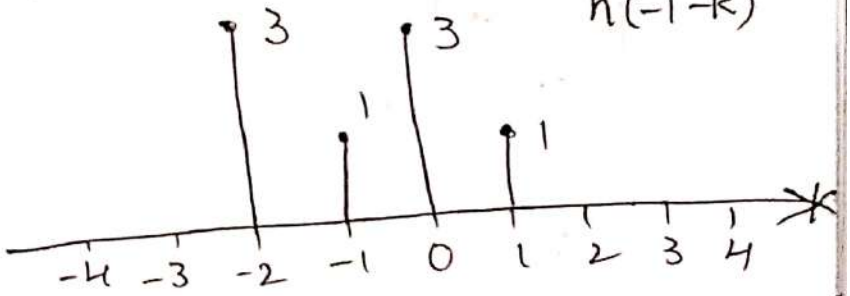


$$y(-2) = 3(3) + 4(1) = 9 + 4$$

$$\boxed{y(-2) = 13}$$

At n = -1 :-

$$y(-1) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(-1-k)$$



$$y(-1) = 3(1) + 4(3) + 2(1)$$

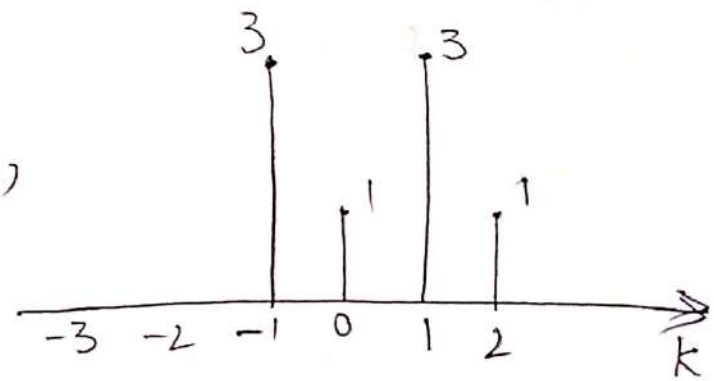
$$= 3 + 12 + 2$$

$$\boxed{y(-1) = 17}$$

At n = 0 :

$$h(0+k) = h(-k)$$

$$y(0) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(-k)$$



$$y(0) = 3(3) + 4(1) + 2(3) + 3(1)$$

$$= 9 + 4 + 6 + 3$$

$$\boxed{y(0) = 22}$$

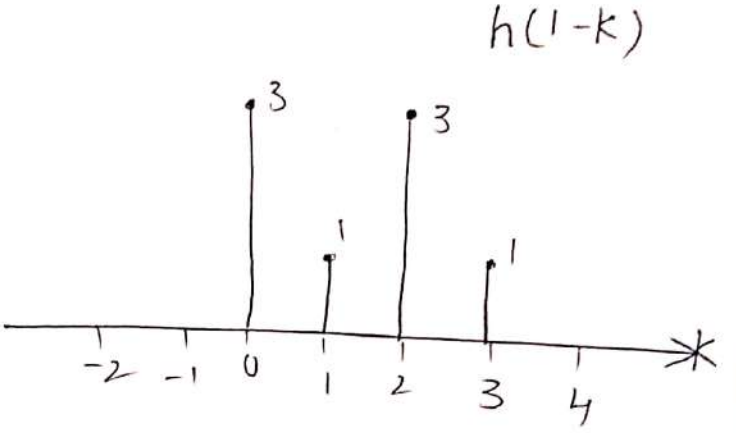
At $n=1$:

$$y(1) = \sum_{k=-\infty}^{\infty} x(k).h(1-k)$$

$$= 4(3) + 2(1) + 3(3) + 0(1)$$

$$= 12 + 2 + 9$$

$$\boxed{y(1) = 23}$$



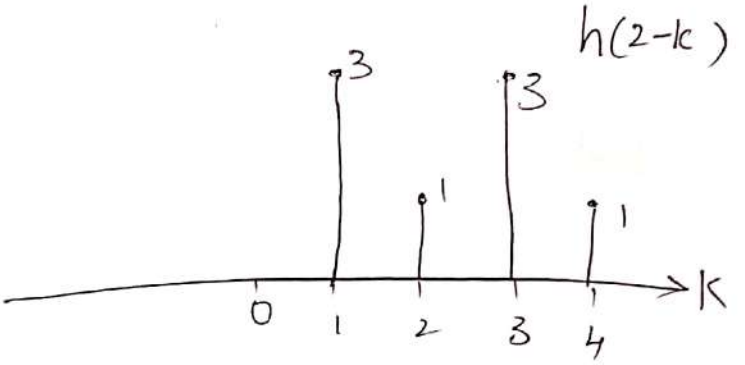
At $n=2$:

$$y(2) = \sum_{k=-\infty}^{\infty} x(k).h(2-k)$$

$$= 2(3) + 3(1)$$

$$y(2) = 6 + 3 = 9$$

$$\boxed{y(2) = 9}$$

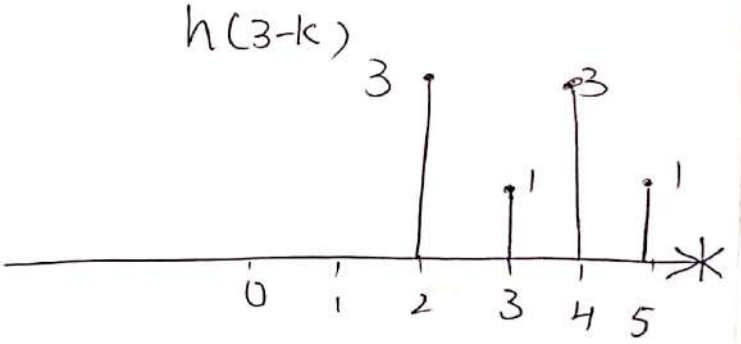


At $n=3$;

$$y(3) = \sum_{k=-\infty}^{\infty} x(k).h(3-k)$$

$$= 3(3)$$

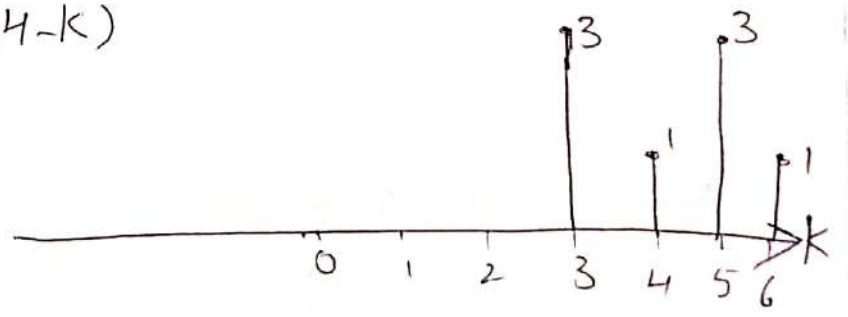
$$\boxed{y(3) = 9}$$



At $n=4$:

$$y(4) = \sum_{k=-\infty}^{\infty} x(k).h(4-k)$$

$$\boxed{y(4) = 0}$$



$$\therefore y(n) = \{3, 13, 17, 22, 23, 9, 9\}$$

Verification:-

$x(n) = \{3, 4, 2, 3\}$ Sum = 12

$h(n) = \{1, 3, 1, 3\}$ Sum = 8

$12 \times 8 = 96$

$y(n) = \{3, 13, 17, 22, 23, 9, 9\}$ Add = 96

$y(n) = x(n) * h(n)$

$\boxed{96 = 12 \times 8}$

Note:-

$N_1 = 4$

$N_2 = 4$

$N = N_1 + N_2 - 1 \Rightarrow 8 - 1 = 7 \Rightarrow \boxed{N = 7}$

If $x(n)$ consists of N_1 samples and $h(n)$ consists of N_2 samples. Then the no. of samples in the output sequence is $N = N_1 + N_2 - 1$

Tabular Method:-

$x(n) = \{3, 4, 2, 3\}$

$h(n) = \{1, 3, 1, 3\}$

$y(n) = \{3, 13, 17, 22, 23, 9, 9\}$

h(n) \ x(n)	3	4	2	3
1	3	4	2	3
3	9	12	6	9
1	3	4	2	3
3	9	12	6	9

Matrix Method:-

Dimensions:-

$$x(n) = [N_1 + N_2 - 1] \times N_1 \longrightarrow 7 \times 4$$

Rows column

$$h(n) = N_1 \times 1 \longrightarrow 4 \times 1$$

Row column

$$x(n) = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 2 & 4 & 3 & 0 \\ 3 & 2 & 4 & 3 \\ 0 & 3 & 2 & 4 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}_{7\times4}, \quad h(n) = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 3 \end{bmatrix}_{4\times1}$$

$$y(n) = x(n) * h(n)$$

$$y(n) = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 2 & 4 & 3 & 0 \\ 3 & 2 & 4 & 3 \\ 0 & 3 & 2 & 4 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}_{7\times4} * \begin{bmatrix} 1 \\ 3 \\ 1 \\ 3 \end{bmatrix}_{4\times1}$$

$$3(1) + 0(3) + 0(1) + 0(3) = 3$$

$$4(1) + 3(3) + 0(1) + 0(3) = 13$$

$2(1)+4(3)+3(1)+0(3)=17$

$3(1)+2(3)+4(1)+3(3)=22$

$0(1)+3(3)+2(1)+4(3)=23$

$0(1)+0(3)+3(1)+2(3)=9$

$0(1)+0(1)+0(1)+3(3)=9$

$\therefore y(n)=\{3, 13, 17, \underset{\uparrow}{22}, 23, 9, 9\}$

(2) $x(n)=\{3, 2, 1, 2\}$, $h(n)=\{1, \underset{\uparrow}{2}, 1, 2\}$ & $y(n)=?$

A. Graphical Method:-

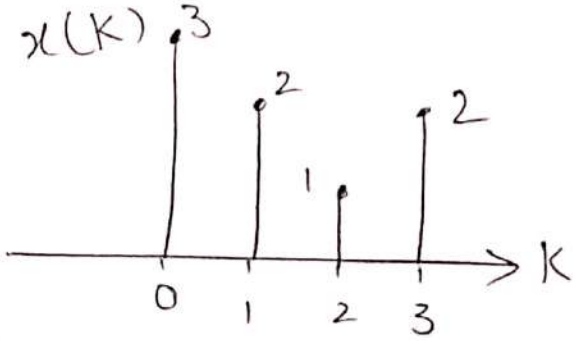
Step 1:- Given, $x(n)=\{3, 2, 1, 2\}$ $n_1=0$

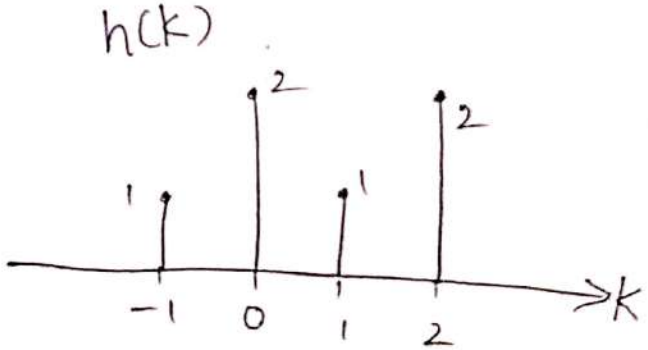
$h(n)=\{1, \underset{\uparrow}{2}, 1, 2\}$ $n_2=-1$

$n=n_1+n_2$

$\boxed{n=-1}$

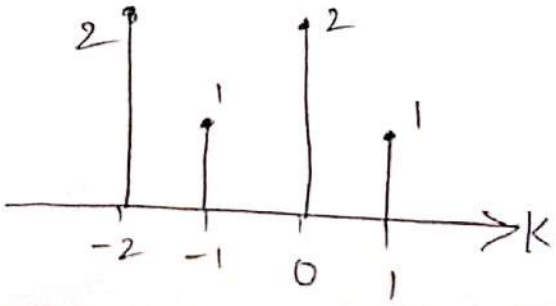
Step 2:- Express the $h(n)$ & $x(n)$ in terms of K.



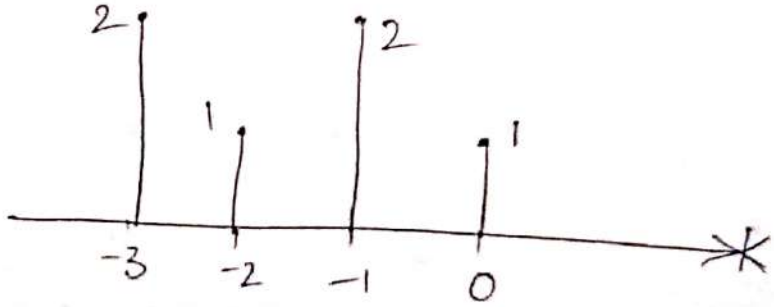


Step 3:-

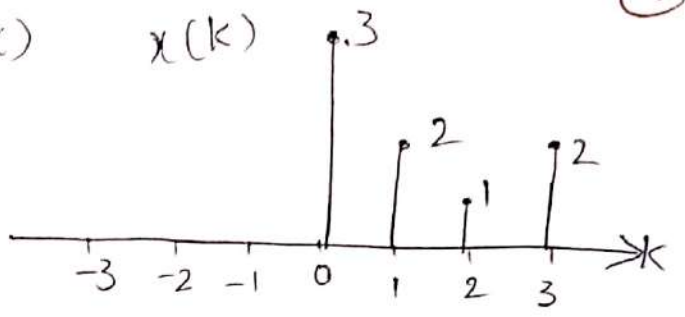
h(-k)



h(n-k) = h(-1-k)



Step 4:- $y(n) = \sum_{k=-\infty}^{\infty} x(k).h(n-k)$



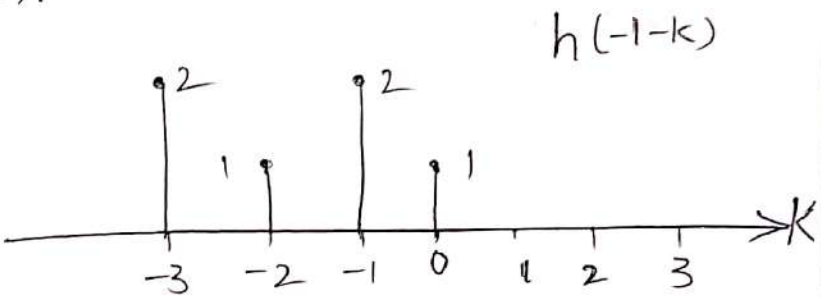
At $n=-1$

$$y(-1) = \sum_{k=-\infty}^{\infty} x(n).h(n-k)$$

$$= \sum_{k=-\infty}^{\infty} x(-1).h(-1-k).$$

$$= 1(3)$$

$$\boxed{y(-1) = 3}$$

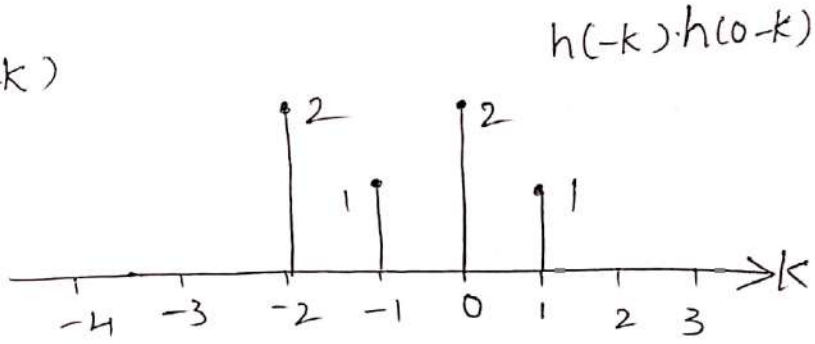


At $n=0$

$$y(0) = \sum_{k=-\infty}^{\infty} x(n).h(0-k)$$

$$= 3(2) + 2(1)$$

$$\boxed{y(0) = 8}$$

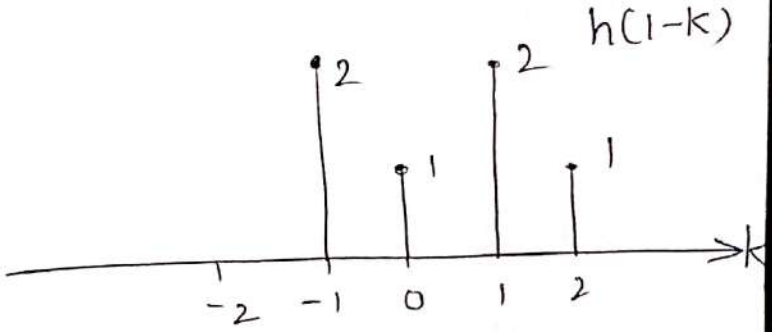


At $n=1$

$$y(1) = \sum_{k=-\infty}^{\infty} x(n).h(1-k)$$

$$= 3(1) + 2(2) + 1(1)$$

$$\boxed{y(1) = 8}$$

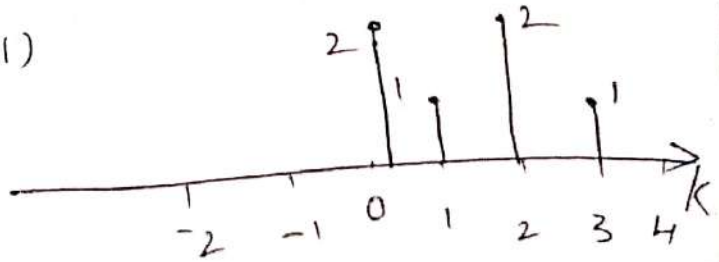


At $n=2$

$$y(2) = \sum_{k=-\infty}^{\infty} x(n).h(2-k)$$

$$= 3(2) + 2(1) + 1(2) + 2(1)$$

$$\boxed{y(2) = 12}$$



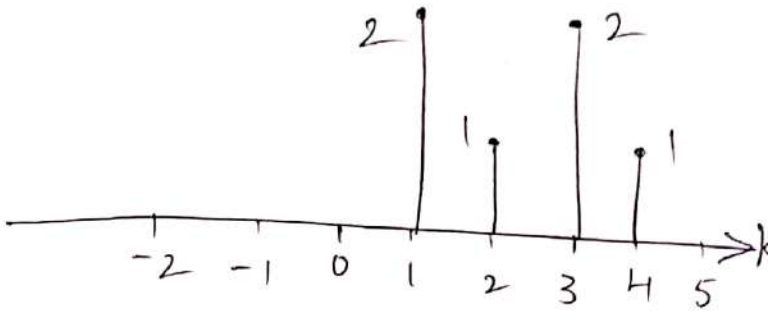
At $n=3$

$$y(3) = \sum_{k=-\infty}^{\infty} x(n) \cdot h(3-k)$$

$$= 2(2) + 1(1) + 2(2)$$

$$\boxed{y(3) = 9}$$

$h(3-k)$

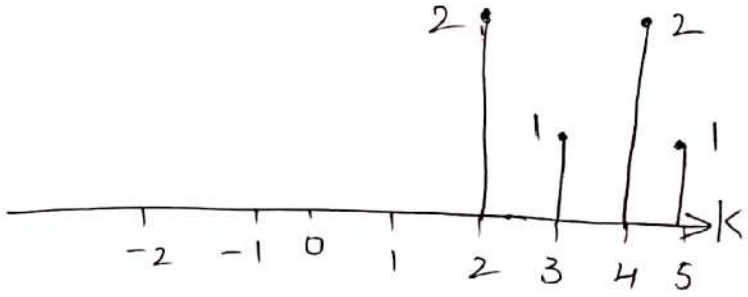


At $n=4$

$$y(4) = \sum_{k=-\infty}^{\infty} x(n) \cdot h(4-k)$$

$$= 1(2) + 2(1)$$

$$\boxed{y(4) = 4}$$

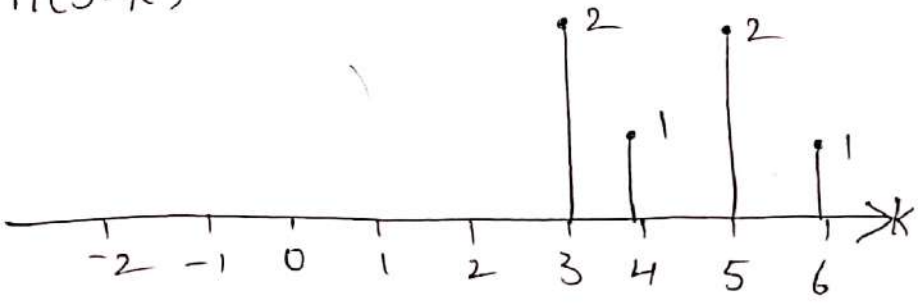


At $n=5$

$$y(5) = \sum_{k=-\infty}^{\infty} x(n) \cdot h(5-k)$$

$$= 2(2)$$

$$\boxed{y(5) = 4}$$



Verification:

$x(n) = \{3, 2, 1, 2\}$ Sum = 8

$h(n) = \{1, 2, 1, 2\}$ Sum = 6

$8 \times 6 = 48$

$y(n) = 48$

$y(n) = \{3, 8, 8, 12, 9, 4, 4\}$.

$N_1 = 4$; $N_2 = 4$.

$N = N_1 + N_2 - 1$

$N = 4 + 4 - 1 = 8 - 1$

$\therefore N = 7$

Tabular Method:-

$x(n) = \{3, 2, 1, 2\}$, $h(n) = \{1, 2, 1, 2\}$

h(n) \ x(n)	3	2	1	2
1	3	2	1	2
2	6	4	2	4
1	3	2	1	2
2	6	4	2	4

$\therefore y(n) = \{3, 8, 8, 12, 9, 4, 4\}$

Matrix Method:-

Dimensions:- $x(n) = (N_1 + N_2 - 1) \times N_1$

$= 7 \times 4$

$h(n) = N_1 \times 1$

$= 4 \times 1$

$$x(n) = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 2 & 1 & 2 & 3 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}_{7 \times 4} , \quad h(n) = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}_{4 \times 1}$$

$y(n) = x(n) * h(n).$

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 2 & 1 & 2 & 3 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}_{7\times4} * \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}_{4\times1}$$

$3(1)+0(2)+0(1)+0(2) = 3$

$2(1)+3(2)+0(1)+0(2) = 8$

$1(1)+2(2)+3(1)+0(2) = 8$

$2(1)+1(2)+2(1)+3(2) = 12$

$0(1)+2(2)+1(1)+2(2) = 9$

$0(1)+0(2)+2(1)+1(2) = 4$

$0(1)+0(2)+0(1)+2(2) = 4$

$\therefore y(n) = \{3, 8, 8, 12, 9, 4, 4\}$

Circular Convolution:-

If $x(n)$ have "L" no. of samples and $h(n)$ has "M" no. of samples then the circular convolution of $x(n)$ and $h(n)$ has "N" no. of samples.

where, $\boxed{N = Max(L, M)}$

Concentric circle Method:-

(1) Locate samples of $x(n)$ has equally spaced points around the outer circle in anti-clockwise direction.

(2) Considering the starting point of $x(n)$. Locate "M" samples of $h(n)$ has equally spaced points around the inner circle in clockwise direction.

(3) Multiply the corresponding samples on the two circles and sum the products to produce the output.

(4) Rotate the inner circle by 1 sample in anti-clockwise direction and go to step-3 to obtain the next sample of the output.

(5) Repeat step-4. until the inner circle 1st sample lines up with the 1st sample of outer circle once again.

* Find the circular convolution of $x(n)$ & $h(n)$ given below.

$x(n) = \{1, -1, -2, -3, -1\}$

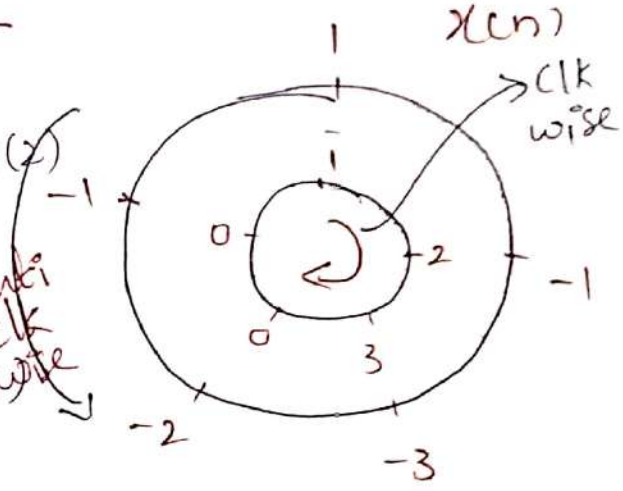
$h(n) = \{1, 2, 3\}$

(A) Given $x(n) = \{1, -1, -2, -3, -1\}$; $N_1 = 5$

$h(n) = \{1, 2, 3\}$

$h(n) = \{1, 2, 3, 0, 0\}$; $N_2 = 5$

Concentric circle Method:-



i) $1(1) + (-1)0 + (-2)(0) + 3(-3) + (-1)(2)$

$= 1 - 9 - 2 \Rightarrow -10$

ii) $2(1) + 1(-1) + (-2)0 + 0(-3) + 3(-1)$

$= 2 - 1 - 3 \Rightarrow -2$

iii) $3(1) + 2(-1) + (-2)1 + (-3)0 + 1(0)$

$= 3 - 2 - 2 = -1$

iv) $0(1) + 3(-1) + (-2)2 + (-3)1 + (-1)0$

$= -3 - 4 - 3 \Rightarrow -10$

v) $0(1) + 0(-1) + (-2)3 \times (-3)2 + 1(-1)$

$= -6 - 6 - 1 = -13$

$\therefore y(n) = \{-10, -2, -1, -10, -13\}$

Matrix Method:-

The sequences $h(n)$ & $x(n)$ consists of "N" samples then the matrix representation for the given sequences are,

$x(n) = N_1 \times N_2 \Rightarrow 5 \times 5$

$h(n) = N_1 \times 1 \Rightarrow 5 \times 1$

Given $x(n) = \{1, -1, -2, -3, -1\}$

$h(n) = \{1, 2, 3, 0, 0\}$

$$y(n) = x(n) \textcircled{+} h(n)$$

$$x(n) = \begin{bmatrix} 1 & -1 & -3 & -2 & -1 \\ -1 & 1 & -1 & -3 & -2 \\ -2 & -1 & +1 & -1 & -3 \\ -3 & -2 & -1 & 1 & -1 \\ -1 & -3 & -2 & -1 & 1 \end{bmatrix}_{5\times5}, \quad h(n) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}_{5\times1}$$

$$y(n) = \begin{bmatrix} 1 & -1 & -3 & -2 & -1 \\ -1 & 1 & -1 & -3 & -2 \\ -2 & -1 & 1 & -1 & -3 \\ -3 & -2 & -1 & 1 & -1 \\ -1 & -3 & -2 & -1 & 1 \end{bmatrix} \textcircled{+} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 1(1) + 2(-1) + 3(-3) + 0 + 0 = -10$

$\Rightarrow 1(-1) + 2(1) + 3(-1) + 0 + 0 = -2$

$\Rightarrow 1(-2) + 2(-1) + 3(1) + 0 + 0 = -1$

$\Rightarrow 1(-3) + 2(-2) + 3(-1) + 0 + 0 = -10$

$\Rightarrow 1(-1) + 2(-3) + 3(-2) + 0 + 0 = -13$

$\therefore y(n) = \{-10, -2, -1, -10, -13\}$

$$y(n) = \begin{bmatrix} -10 \\ -2 \\ -1 \\ -10 \\ -13 \end{bmatrix}$$

$=$

Correlation:-

Correlation is used to compare two sig's. It is a measure of similarity between the signals and is found using a process similar to convolution.

* Correlation is of two types.
1. Auto correlation
2. Cross correlation.

Auto Correlation:-

The auto correlation of a sequence is the correlation of the sequence with it self. It gives a measure of similarity between a sequence and its shifted version.

The auto correlation of a sequence $x(n)$ is represented as "$R_{xx}(n)$".

$$\therefore R_{xx}(n) = \sum_{K=-\infty}^{\infty} x(K).x(K-n) \quad \text{--- (1)}$$

$$R_{xx}(n) = \sum_{K=-\infty}^{\infty} x(K+n).x(K) \quad \text{--- (2)}$$

$$R_{xx}(n) = x(n).x(-n).$$

* Find the correlation of the given sequence

$x(n) = \{1, 0, \underset{\uparrow}{2}, 3\}$

* Given $x(n) = \{1, 0, 2, 3\}$ $n_1 = -2$

$x(-n) = \{3, 2, 0, 1\}$ $n_2 = -1$

$R_{xx} = x(n) \cdot x(-n)$

$x(-n)$ \ $x(n)$	1	0	2	3
3	3	0	6	9
2	2	0	4	6
0	0	0	0	0
1	1	0	2	3

$R_{xx}(n) = \{3, 2, 6, 14, 6, 2, 3\}$

(2) Cross correlation:-

The cross correlation is a measure of similarity between the pair of two sequences $x(n)$ & $y(n)$ and is given by "$R_{xy}(n)$".

$$R_{xy}(n) = \sum_{K=-\infty}^{\infty} x(K) \cdot y(K-n)$$

(a)

$$R_{xy}(n) = \sum_{K=-\infty}^{\infty} x(K-n) \cdot y(K)$$

(b)

$$R_{xy}(n) = \sum_{K=-\infty}^{\infty} x(K) \cdot y(K-n)$$

(a)

$$R_{xy}(n) = \sum_{K=-\infty}^{\infty} x(K+n) \cdot y(K)$$

(b)

$$R_{xy}(n) = x(n) \cdot y(-n).$$

* Find the cross correlation of the given sequences $x(n) = \{1, 1, 2, 2\}$ and $y(n) = \{1, 2, 3, 4\}$.

A. Given $x(n) = \{1, 1, 2, 2\}$, $y(n) = \{1, 2, 3, 4\}$

$$R_{xy}(n) = x(n) * y(-n)$$

$$y(-n) = \{4, 3, 2, 1\}$$

$$\therefore R_{xy}(n) = x(n) * y(-n)$$

$y(-n)$ \ $x(n)$	1	1	2	2
4	4	4	8	8
3	3	3	6	6
2	2	2	4	4
1	1	1	2	2

$$\therefore R_{xy}(n) = \{4, 7, 13, 17, 16, 6, 2\}$$

Mathematical Equations Governing the Discrete Time system :-

* The response of the discrete time system at any instant of time depends on the present input, past inputs & past outputs.
* Let us consider the response of the system at $n=0$ given by

At $n=0$; $y(0) = F[x(0)]$

At $n=1$; $y(1) = F[y(0), x(1), x(0)]$

At $n=2$; $y(2) = F[y(1), y(0), x(0), x(1), x(2)]$

* In general, at any instant of time "n", the response of the system can be written as

$$y(n) = F[y(n-1), y(n-2) \cdots y(1), y(0), x(1), x(2), \cdots x(n-1), x(n-2)]$$

* For an LTI Sm the response $y(n)$ can be expressed as weighted sum of the dependent terms that can be written as

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) \cdots b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \cdots$$

where a_0, a_1, a_2, b_0, b_1 are constants.

* Practically, the response $y(n)$, at any instant of time "n" depends on "N" number of past outputs, present input and "M" number of past inputs. where, "$M \leq N$".

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k) \longrightarrow (1)$$

Eqn (1) gives linear constant coefficient difference equation governing the input & output relation of an LTI discrete time systems.

$$\therefore \sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \quad (\because a_0 = 1)$$

Analysis of Discrete time LTI system in Z-Domain:-

Z-Transform: The Z-Transform of a given sequence $x(n)$ is defined as $X(z)$.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n} \longrightarrow (1)$$

where z is a complex variable and it is given by $z = r \cdot e^{j\omega}$.

where, r is the radius of a unit circle.

$$X(z) = X(r \cdot e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) \cdot (r \cdot e^{j\omega})^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \cdot r^{-n} \cdot e^{-j\omega n}.$$

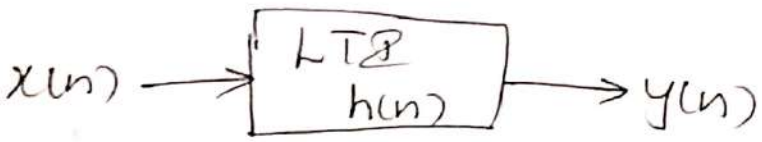
$$X(z) = X(r \cdot e^{j\omega}) = \sum_{n=-\infty}^{\infty} \left[x(n) \cdot r^{-n} \right] e^{-j\omega n} \longrightarrow (2)$$

The equation (2) is called the discrete time F.T. of the input signal $x(n) \cdot r^{-n}$.

If the radius of the unit circle $r = 1$, then the Z-Transform is same as DTFT.

$$X(z) = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n}.$$

Consider a discrete time LTI system with impulse response $h(n)$ as shown.



$y(n) = x(n) * h(n)$

Apply Z-Transform

$$Y(Z) = X(Z) \cdot H(Z).$$

$$\frac{Y(Z)}{X(Z)} = H(Z).$$

where, $H(Z)$ is transfer function.

The frequency response of the system is obtained by substituting $Z = e^{j\omega}$ in $H(Z)$.

if $Z = e^{j\omega}$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

→ Zeros

→ Poles.

$H(e^{j\omega})$ = Frequency Response.

The poles and zeros of the frequency response have an insight of the stm characteristics.

The poles of the system are defined as the values for z, for which $H(Z)$ is infinity and zeros of the stm are the values of the Z for which $H(Z)$ is zero.

An N^{th} order discrete time LTI sys is specified in difference equation as,

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-l)$$

Apply Z-Transform on both sides.

$$\sum_{k=0}^{N} a_k \cdot z^{-k} \cdot Y(z) = \sum_{k=0}^{M} b_k \cdot z^{-k} \cdot X(z).$$

$$\frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k \cdot z^{-k}}{\sum_{k=0}^{N} a_k \cdot z^{-k}}.$$

Stability & Causality :-

* The necessary and sufficient condition for a causal LTI sys to be stable if,

$\sum_{n=0}^{\infty} |H(n)| < \infty$ i.e., the impulse response of the sys must be absolutely summable.

* For a causal LTI sys to be stable all the poles of $H(z)$ must lie inside the unit circle in z-plane.

* Consider an LTI sys. with the sys function $H(z) = \frac{1}{1 - \frac{1}{2} z^{-1}}$. Find the difference equation and determine the stability.

A. Given $H(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$

$$\frac{Y(z)}{X(z)} = \frac{1}{1-\frac{1}{2}z^{-1}}$$

$$Y(z)\left[1-\frac{1}{2}z^{-1}\right] = X(z)$$

$$Y(z) - \frac{1}{2}z^{-1}Y(z) = X(z)$$

Apply Inverse Z-Transform

$$x(n) = y(n) - \frac{1}{2}y(n-1)$$

Stability:-

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$$

$$1-\frac{1}{2}z^{-1} = 0$$

$$z^{-1} = 2$$

$$\frac{1}{z} = 2 \Longrightarrow z = \frac{1}{2} = 0.5$$

∴ The poles of the $H(z)$ is $\frac{1}{2}$ which lies inside the unit circle, hence the system is stable.

Frequency Domain Representation of Discrete Time Signals & Systems:-

* The output response $y(n)$ of any LTI S/m to an input $x(n)$ can be obtain using convolution sum.

$$y(n) = \sum_{k=-\infty}^{\infty} h(k).x(n-k) \longrightarrow (1)$$

* To develop a frequency domain characterization of the Slm, let us excite the Slm with a complex exponential function,

$$x(n) = e^{j\omega n} \longrightarrow (2)$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k) \cdot e^{j\omega(n-k)}$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k) \cdot e^{j\omega n} \cdot e^{-j\omega k}.$$

$$y(n) = e^{j\omega n} \cdot \sum_{k=-\infty}^{\infty} h(k) \cdot e^{-j\omega k}.$$

$$\boxed{y(n) = e^{j\omega n} \cdot H(e^{j\omega})} \longrightarrow (3)$$

$$\text{where, } H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k) \cdot e^{-j\omega k}.$$

$$\left[\because x(n) = e^{j\omega n}, \quad x(n-k) = e^{j\omega(n-k)} \right]$$

* From the above expression we can observe that for an i/p sig $x(n) = e^{j\omega n}$, the o/p of a discrete time LTI Slm is also an exponential sig of same frequency multiplied by a factor $H(e^{j\omega})$.

* The sigs that produce a response which differ from the i/p sig by a complex constant are known as eigen function. The value of complex constant $H(e^{j\omega})$ is called eigen values of the Slm.

* The quantity $(H(e^{j\omega}))$ is called frequency response of the system.

$$H(e^{j\omega}) = H_I(e^{j\omega}) + H_R(e^{j\omega})$$

$$H(e^{j\omega}) = |H(e^{j\omega})| \angle H(e^{j\omega})$$

where, $|H(e^{j\omega})| = H_I^2(e^{j\omega}) + H_R^2(e^{j\omega})$ → Magnitude Response

$$\angle H(e^{j\omega}) = \tan^{-1}\left(\frac{H_I(e^{j\omega})}{H_R(e^{j\omega})}\right) \rightarrow \text{Phase Response.}$$

* Determine & sketch the magnitude & Phase response of $y(n) = \frac{1}{2}[x(n) + x(n-2)]$

A. Given $y(n) = \frac{1}{2}[x(n) + x(n-2)]$

Apply Fourier Transform

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y(n) \cdot e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\frac{1}{2}[x(n) + x(n-2)]\right] e^{-j\omega n}$$

$$= \frac{1}{2}\left[\sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n} + \sum_{n=-\infty}^{\infty} x(n-2)\, e^{-j\omega n}\right]$$

$$= \frac{1}{2}\left[X(e^{j\omega}) + e^{-2j\omega} X(e^{j\omega})\right]$$

$$= \frac{1}{2}\left[X(e^{j\omega})(1 + e^{-2j\omega})\right]$$

$$\frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{2}\left[1 + e^{-2j\omega}\right]$$

$$H[e^{j\omega}] = \frac{1}{2}\left[1 + e^{-2j\omega}\right]$$

$$\boxed{H(e^{j\omega}) = \frac{1}{2}\left[1 + \cos 2\omega - j \sin 2\omega\right]} \rightarrow (1)$$

Magnitude Response:-

$$|H(e^{j\omega})| = \frac{1}{2}\sqrt{(1+\cos 2\omega)^2 + (\sin 2\omega)^2}$$

$$= \frac{1}{2}\left[\sqrt{1+\cos^2 2\omega + 2\cos 2\omega + \sin^2 2\omega}\right]$$

$$= \frac{1}{2}\left[\sqrt{2+2\cos 2\omega}\right]$$

$$= \frac{1}{2}\left[\sqrt{2(1+\cos 2\omega)}\right]$$

$$= \frac{1}{2}\left[\sqrt{2(2\cos^2\omega)}\right]$$

$$= \frac{1}{2}\sqrt{4\cos^2\omega}$$

$$|H(e^{j\omega})| = \frac{1}{2}\left(\sqrt{4}\cdot\sqrt{\cos^2\omega}\right)$$

$$|H(e^{j\omega})| = \frac{1}{2}\cdot(2\cdot\cos\omega)$$

$$\boxed{|H(e^{j\omega})| = \cos\omega} \rightarrow (2)$$

Phase Response:-

$$\angle H(e^{j\omega}) = \tan^{-1}\left(\frac{-\sin 2\omega}{1+\cos 2\omega}\right)$$

$$= \tan^{-1}\left(\frac{-\sin 2\omega}{2\cos^2\omega}\right)$$

$$= \tan^{-1}\left(\frac{-2\sin\omega\cdot\cos\omega}{2\cos^2\omega}\right)$$

$$= \tan^{-1}(-\tan\omega) = -\omega.$$

$$\angle H(e^{j\omega}) = -\omega \quad \text{for} \quad |H(e^{j\omega})| > 0$$

$$= -\omega + \pi \quad \text{for} \quad |H(e^{j\omega})| < 0$$

(3)

ω	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	π
$H(e^{j\omega})$	1	0.812	0.707	0.5	0	-0.5	-0.707	-1
$\lvert H(e^{j\omega})\rvert$	1	0.812	0.707	0.5	0	0.5	0.707	1
$\angle H(e^{j\omega})$	0	$-\pi/6$	$-\pi/4$	$-\pi/3$	$-\pi/2$	$\pi/3$	$\pi/4$	0

Z-TRANSFORM

PART-B

Transform technique are an important tool in the analysis of Sigs & Slms.

* The Laplace Transforms are popularly used for analysis of continuous time Sigs & Slms. Slly Z-Transforms plays an important role in the analysis of representation of DT Sigs & Slm.
* The Z-Transforms provides a method for the analysis of discrete time Sigs & Slms in the frequency domain, which is generally more efficient than its time domain analysis.
* The Z-Transform of x(n) will convert the time domain signal x(n) to Z-domain signal X(z), where the signal becomes a function of complex variable "z".

The complex variable Z is defined as,

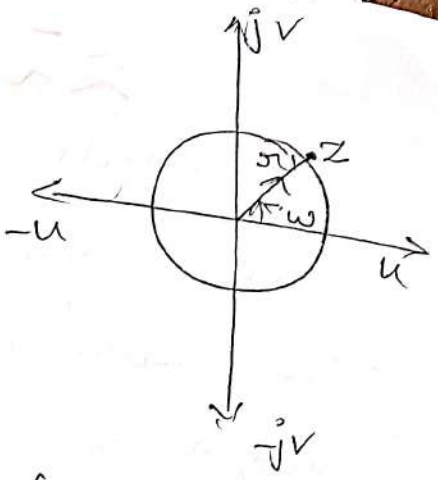
$$Z = u + vj = r.e^{j\omega}.$$

where, u = real part of Z.

v = Imag part of Z.

$r = \sqrt{u^2+v^2}$ = Mag. of z.

$\omega = \tan^{-1}\frac{v}{u}$ = phase / Argument of z.



The 'u' and 'v' takes value from $-\infty$ to ∞.

fig: Z-plane.

* A two dimensional complex plane with values of 'u' on horizontal axis and values of 'v' on vertical axis as shown in above fig. is called Z-plane.

* A circle with radius 'r_1' in Z-plane represents all values of z_1 having same magnitude r_1 with variable phase ω_1, where $\omega_1 = 0$ to 2π.

History :-

A transform of a sampled signal / sequence was defined in 1947 by W. Hurewicz as

$$Z[f(KT)] = \sum_{k=0}^{\infty} f(KT) z^{-1}$$

which was later denoted in 1952 as Z-Transform by a sampled control group at Columbia University by prof. R. John. Raggazini & some others

Defination :—

Let, $x(n)$ = Discrete time sig

$X(z)$ = Z-Transform of $x(n)$.

The Z-Transform of a discrete time signal, $x(n)$ is defined as,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \quad \longrightarrow (1)$$

; where, z is a complex variable.

The Z-Transform of $x(n)$ is symbolically denoted as, $Z\{x(n)\}$; where, Z is the operator that represt Z-transform.

$$\therefore X(z) = Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

The time index 'n' is defined for both +ve & -ve values, if the sig $x(n)$ is one-sided sig [i.e, $x(n)$ is defined only for +ve value of n]. then the Z-Transform is called one-sided Z-Transform.

∴ The one-sided Z-Transform of $x(n)$ is defined as

$$X(z) = Z\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}. \quad \longrightarrow (2)$$

The computation of $X(z)$ involves summation of infinite terms which are functions of z.

* Hence it is possible that the infinite series may not converge to finite value for certain values of z.
* Therefore for every $X(z)$ there will be a set of values of z for which $X(z)$ can be computed.
* Such a set of values will lie in a particular region of z-plane and this region is called Region of convergence (ROC) of $X(z)$.

Inverse Z-Transform:-

Let, $x(z)$ be Z-Transform of $x(n)$. Now the signal $x(n)$ can be uniquely determined from $X(z)$ and its region of convergence (ROC).

The Inverse Z-Transform of $X(z)$ is defined as,

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) \cdot z^{n-1} dz \longrightarrow (1)$$

The Inverse Z-Transform of $X(z)$ is symbolically denoted as,

$Z^{-1}\{X(z)\}$; where, Z^{-1} is the operator that represents the inverse Z-Transform.

$$\therefore x(n) = Z^{-1}\{X(z)\} = \frac{1}{2\pi j} \oint_C X(z) \cdot z^{n-1} dz$$

we also refer $x(n)$ and $X(z)$ as a Z-Transform pair and this relation is expressed as,

$$x(n) \underset{Z^{-1}}{\overset{Z}{\rightleftharpoons}} X(z)$$

Geometric series :-

The Z-transform of a discrete time sig involves convergence of geometric series. The following are the two geometric series sum formula for evaluating Z-Transforms.

(1) Infinite geometric series sum formula.

If "c" is a complex constant and $0 < |c| < 1$, then

$$\sum_{n=0}^{\infty} c^n = \frac{1}{1-c} \longrightarrow (1)$$

(2) Finite geometric series sum formula.

If "c" is a complex constant and,

when $c \neq 1$, $\sum_{n=0}^{N-1} c^n = \frac{1-c^N}{1-c} = \frac{c^N - 1}{c-1}$

(or)

$$c^n = \frac{c^{N+1} - 1}{c-1}$$

when $c=1$, $\sum_{n=0}^{N-1} c^n = N$ (or) $\sum_{n=0}^{N} c^n = N+1$

Note:- The infinite geometric series sum formula requires that the magnitude of 'c' be strictly less than unity, but the finite geometric series sum formula is valid for any value of 'c'.

Region of Convergence:-

Z-Transform is an infinite power series, it exists only for those values of z for which the series converges.

The region of convergence (ROC) of $x(z)$ is the set of all values of z, for which $x(z)$ attains a finite value.

The ROC for the signals has 6 types.

Case (i):- Finite duration, right-sided (causal) sig:

Let, $x(n)$ be a finite duration signal with N-samples, defined in the range $0 \le n \le (N-1)$.

$\therefore x(n) = \{x(0), x(1), x(2) \ldots x(N-1)\}$

Now, the Z-Transform of $x(n)$ is,

$$X(z) = \sum_{n=0}^{N-1} x(n) \cdot z^{-n}.$$

$$= x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots + x(N-1)z^{-(N-1)}$$

$$= x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \cdots + \frac{x(N-1)}{z^{N-1}}$$

In the above summation, when $z=0$, all the terms except the first term become infinite.

Hence the $X(z)$ exists for all values of z, except $z=0$.

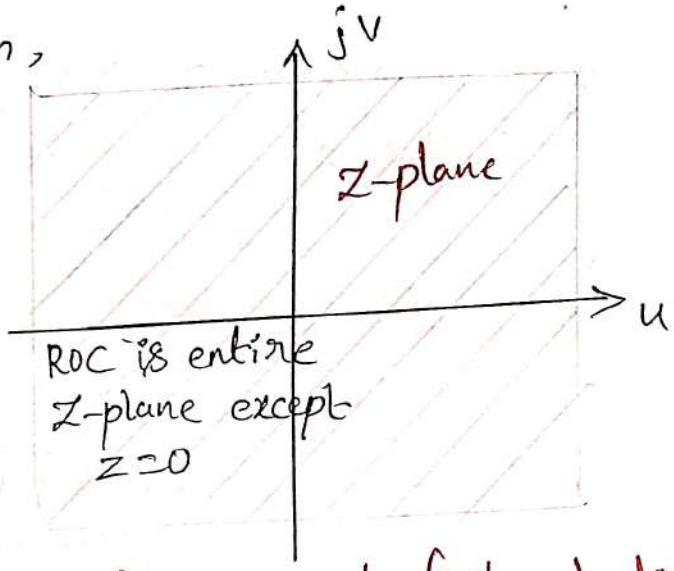


fig: ROC of finite duration causal signal.

Therefore the ROC of finite duration right-sided (or causal sig) is entire Z-plane except $z=0$.

Case (ii): Finite duration, left sided (anti-causal) sig:-

Let, $x(n)$ be a finite duration signal with N-samples, defined in the range $-(N-1) \le n < 0$.

$$\therefore x(n) = \{x(-(N-1)), \cdots, x(-2), x(-1), x(0)\}.$$

Now, the Z-Transform of $x(n)$ is,

$$X(z) = \sum_{n=-(N-1)}^{0} x(n) \cdot z^{-n}.$$

$$\Rightarrow x(-(N-1))\, z^{(N-1)} + \cdots + x(-2)z^2 + x(-1)z + x(0).$$

In the above summation, when $z=\infty$, all the terms except the last term become an infinite. Hence the $X(z)$ is exists for all values of z, except $z=\infty$.

∴ The ROC of $X(z)$ is entire z-plane, except $z=\infty$.

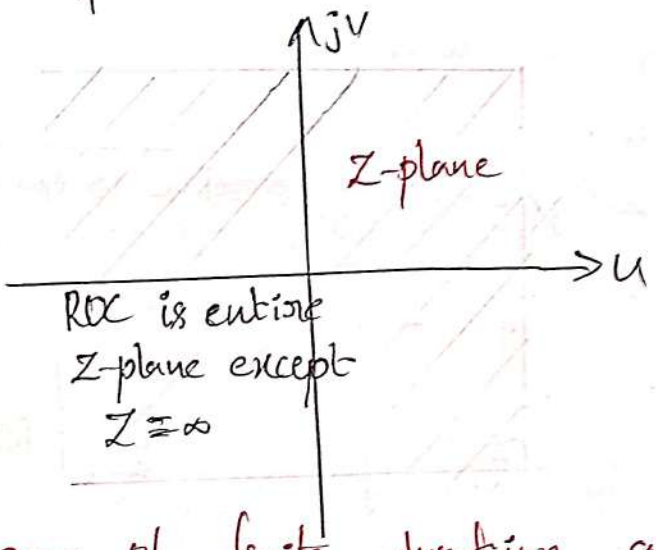


fig: ROC of finite duration anticausal sig.

Case(iii): Finite duration, two-sided (Non-causal) sig:-

Let, $x(n)$ be a finite duration sig with N-samples defined in the range $-M \le n \le +M$, where $M = \frac{N-1}{2}$.

$\therefore x(n) = \{x(-M), \ldots x(-2), x(-1), x(0), x(1), x(2), \ldots x(M)\}$.

Now, the Z-Transform of $x(n)$ is,

$$X(z) = \sum_{n=-M}^{+M} x(n).z^{-n}.$$

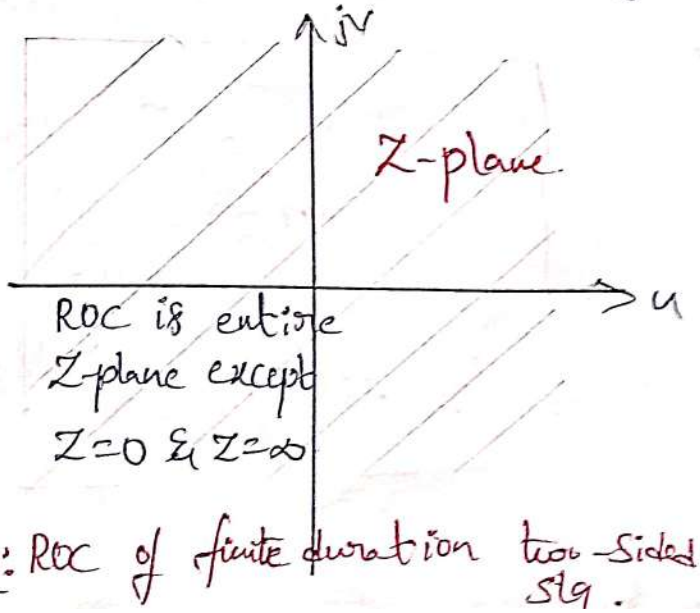


fig: ROC of finite duration two-sided sig.

$$= x(-M)z^{M} + \cdots + x(-2)z^{2} + x(-1)z + x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots + x(M)z^{-M}.$$

$$= x(-M)z^{M} + \cdots + x(-2)z^{2} + x(-1)z + x(0) + \frac{x(1)}{z} + \frac{x(2)}{z} + \cdots + \frac{x(M)}{z^{M}}.$$

In the above summation, when $z=0$, the terms with negative power of z attain infinity and when $z=\infty$, the term with positive power of z attain infinity.

Hence $X(z)$ converges for all values of z, except $z=0$ and $z=\infty$. $\therefore$ ROC is entire Z-plane except $z=0$ & $z=\infty$.

Case (iv):- Infinite duration, right-sided (causal) sig:-

Let, $x(n) = r_1^{n}$; $n \geq 0$

Now, the Z-Transform of $x(n)$ is

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\cdot z^{-n} = \sum_{n=0}^{\infty} r_1^{n}\cdot z^{-n} = \sum_{n=0}^{\infty} (r_1 z^{-1})^{n}.$$

If, $0 < |r_1 z^{-1}| < 1$, then $\sum_{n=0}^{\infty} (r_1 z^{-1})^{n} = \frac{1}{1-r_1 z^{-1}}$

$$\therefore X(z) = \frac{1}{1-r_1 z^{-1}} = \frac{1}{1-\frac{r_1}{z}} = \frac{1}{\frac{z-r_1}{z}} = \frac{z}{z-r_1}$$

Here the condition to be satisfied for the convergence of $X(z)$ is, $0 < |r_1 z^{-1}| < 1$

$$\therefore |r_1 z^{-1}| < 1 \Longrightarrow \frac{|r_1|}{|z|} < 1 \Longrightarrow |z| > |r_1|$$

The term $|z_1|$ represents a circle of radius z_1 in Z-plane. From the above analysis, we can say that, X(z) converges for all points external to the circle of radius z_1 in Z-plane.

∴ The ROC of X(z) is exterior of the circle of radius 'z_1' in Z-plane.

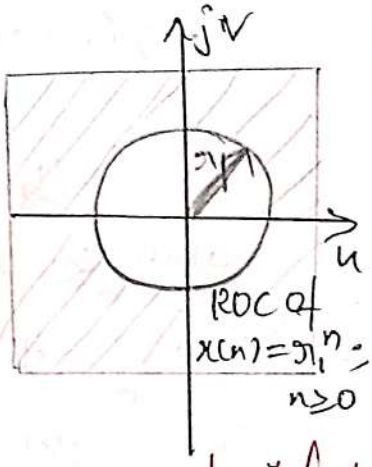


fig: ROC of infinite duration right sided sig.

Case (v):- Infinite duration, left sided (anticausal) sig.

Let, $x(n) = z_2^n$; $n \le 0$.

Now, the Z-transform of x(n) is

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n} = \sum_{n=-\infty}^{0} z_2^n z^{-n} = \sum_{n=0}^{\infty} (z_2^{-n} z^n)$$

$$\Rightarrow \sum_{n=0}^{\infty} (z_2^{-1} z)^n$$

If, $0 < |z_2^{-1} z| < 1$, then

$$\sum_{n=0}^{\infty} (z_2^{-1} z)^n = \frac{1}{1 - z_2^{-1} z}$$

$$\therefore X(z) = \frac{1}{1 - z_2^{-1} z} = \frac{1}{1 - \frac{z}{z_2}} = \frac{1}{\frac{z_2 - z}{z_2}} = \frac{z_2}{z_2 - z}$$

$$\Rightarrow \frac{-z_2}{z - z_2}$$

Here the condition to be satisfied for the convergence of $X(z)$ is,

$$0 < |r_2^{-1} z| < 1$$

$$\therefore |r_2^{-1} z| < 1 \Rightarrow \frac{|z|}{|r_2|} < 1 \Rightarrow |z| < |r_2|$$

The term $|r_2|$ represents a circle of radius r_2 in z-plane as shown in fig. From the above analysis we can say that $X(z)$ converges for all points internal to the circle of radius r_2 in z-plane. Therfore, the ROC of $X(z)$ is interior of the circle of radius r_2 as shown in fig.

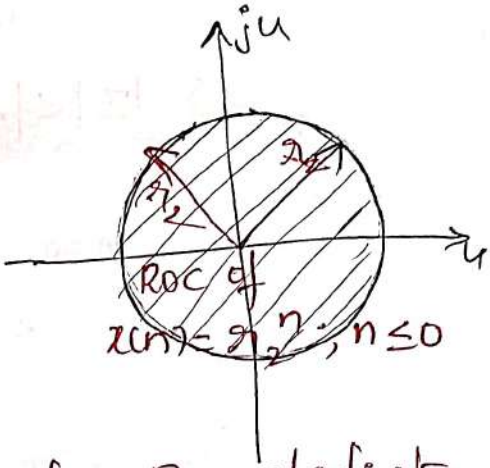


fig:- ROC of infinite duration left-sided sig

case (vi):- Infinite duration, two-sided (non-causal) signal:-

let, $x(n) = r_1^n u(n) + r_2^n u(-n)$

Now, the Z-Transform of $x(n)$ is,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{0} r_2^n z^{-n} + \sum_{n=0}^{\infty} r_1^n z^{-n}$$

$$= \sum_{n=0}^{\infty} r_2^{-n} z^n + \sum_{n=0}^{\infty} r_1^n z^{-n}$$

$$= \frac{1}{1 - r_2^{-1} z} + \frac{1}{1 - r_1 z^{-1}}$$

The term $\sum_{n=0}^{\infty} (r_2^{-1} z)$ converges if,

$0 < |r_2^{-1} z| < 1$

$\therefore |r_2^{-1} z| < 1 \Rightarrow \frac{|z|}{|r_2|} < 1$

$\Rightarrow |z| < |r_2|$.

Infinite geometric series sum formula

$\sum_{n=0}^{\infty} c^n = \frac{1}{1-c}$ if, $0 < |c| < 1$.

Using infinite geometric series sum formula

if, $0 < |r_2^{-1} z| < 1$, & $0 < |r_1 z^{-1}| < 1$.

The term $\sum_{n=0}^{\infty} (r_1 z^{-1})$ converges if., $0 < |r_1 z^{-1}| < 1$

$\therefore |r_1 z^{-1}| < 1 \Rightarrow \frac{|r_1|}{|z|} < 1 \Rightarrow |z| > |r_1|$

The term $|r_2|$ represents a circle of radius r_2 and $|r_1|$ represents a circle of radius r_1 in z-plane.

If $|r_2| > |r_1|$ then there will be a region btw 2 circles as shown in in bove.

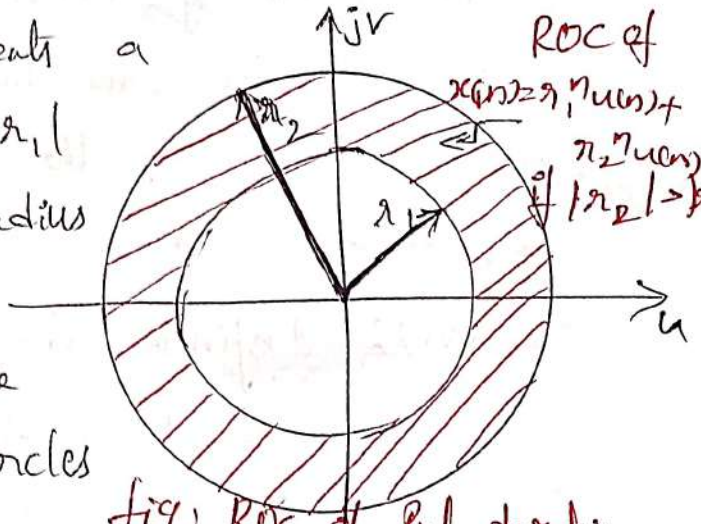


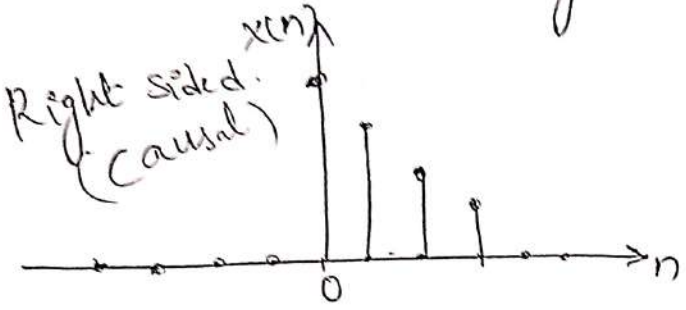
fig: ROC of Inf. duration 2 Sided Sig.

Now the X(z) will converge for all points in the region btw two circles (because of the first term of X(z) converges for $|z| < |r_2|$ and the second term of X(z) converges for $|z| > |r_1|$.

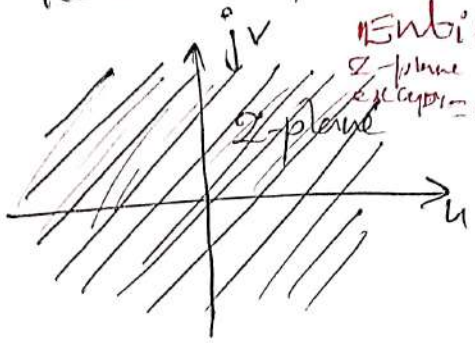
Hence the ROC is the second term of region btw two circles of radius r_1 & r_2 as shown in fig.

Characteristic Families of Sig & Corresponding ROC:-

Finite duration Sig



ROC in Z-plane.



* Determine the Z-Transform and their ROC of the following discrete time signals.

(a) $x(n) = \{3, 4, 2, 7\}$
$\uparrow$

Let $x(n) = \{3, 4, 2, 7\}$
$\uparrow$

i.e. $x(0) = 3$, $x(1) = 4$, $x(2) = 2$, $x(3) = 7$;

$x(n) = 0$ for $n < 0$ and $n > 3$

By the definition of Z-Transform

$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

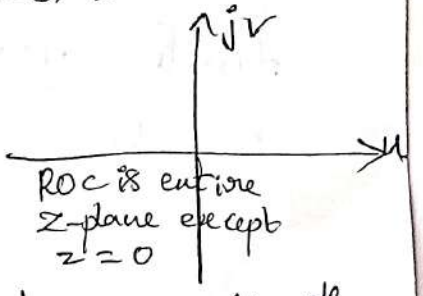
The given sequence is a finite duration sequence defined in the range $n \geq 0$ to 3, hence the limits of summation is changed to $n \geq 0$ to $n \geq 3$.

$$\therefore X(z) = \sum_{n \geq 0}^{3} x(n) z^{-n}.$$

$$= x(0) z^0 + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3}$$

$$= 3 + 4z^{-1} + 2z^{-2} + 7z^{-3}$$

$$= 3 + \frac{4}{z} + \frac{2}{z^2} + \frac{7}{z^3}$$



$\therefore$ In $X(z)$, when $z = 0$, except the first term all other terms will become infinite. Hence $X(z)$ will be finite for all values of z, except $z = 0$. Therefore, the ROC is entire Z-plane except $z = 0$.

(b) $x(n) = \{6, 8, 9, 3\}$
$\uparrow$

i.e. $x(-3)=6$, $x(-2)=8$, $x(-1)=9$, $x(0)=3$, and
$x(n)=0$ for $n<-3$ & for $n>0$.

By the defination of Z-Transform.

$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}.$$

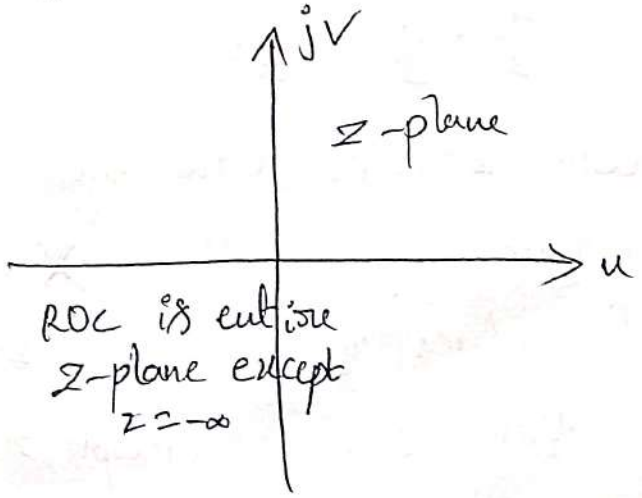
The given sequence is a finite duration sequence defined in the range $n=-3$ to 0, hence the limit of sumation is chaned to $n=-3$ to 0.

$$\therefore X(z) = \sum_{n=-3}^{0} x(n) z^{-n}$$

$$= x(-3) z^3 + x(-2) z^2 + x(-1) z + x(0)$$

$$X(z) = 6z^3 + 8z^2 + 9z + 3$$

In $X(z)$, when $z=\infty$, except last term all other terms become infinite. Hence $X(z)$ will be finite for all values of z, except $z=\infty$. Therefore, the ROC is entire Z-plane except $z=\infty$.



(i) $x(n) = \{2, 4, 6, 8, 10\}$

i.e. $x(-2)=2, x(-1)=4, x(0)=6, x(1)=8, x(2)=10$.

and $x(n)=0$ for $n<-2$ and for $n>2$.

By the definition of Z-Transform

$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

The given sequence is a finite duration sequence defined in the range $n=-2$ to 2, Hence the limits of summation is changed to $n=-2$ to $n=2$

$$\therefore X(z) = \sum_{n=-2}^{2} x(n) \cdot z^{-n}$$

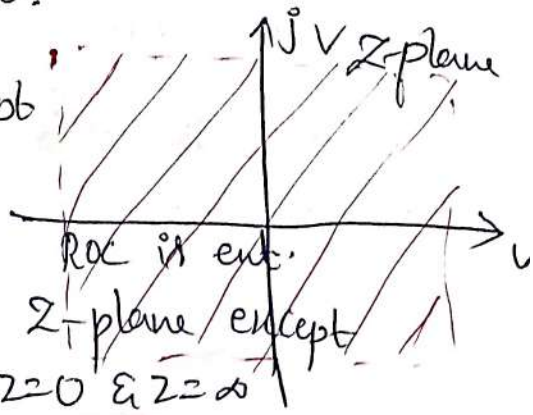
$$= x(-2)z^{2} + x(-1)z^{1} + x(0)z^{0} + x(1)z^{-1} + x(2)z^{-2}$$

$$= 2z^{2} + 4z + 6 + 8z^{-1} + 10z^{-2}$$

$$X(z) = 2z^{2} + 4z + 6 + \frac{8}{z} + \frac{10}{z^{2}}.$$

In $X(z)$, when $z=0$, the terms with negative power of z will become infinite and when $z=\infty$, the terms with positive power of z will become infinite. Hence $X(z)$ will be finite for all values of z except when $z=0$ and $z=\infty$.

$\therefore$ The ROC is entire Z-plane except $z=0$ & $z=\infty$.



* Determine the Z-Transform and their ROC of the following discrete time signals.

(a) $x(n) = u(n)$

Given $x(n) = u(n)$

The $u(n)$ is a discrete time unit step sig, which is defined as,

$$u(n) = 1 \; ; \text{ for } n \geq 0$$
$$0 \; ; \text{ for } n < 0$$

By the definition of Z-Transform,

$$Z\{x(n)\} = X(Z) = \sum_{n=-\infty}^{\infty} x(n) \cdot Z^{-n}$$

$$= \sum_{n=0}^{\infty} u(n) \cdot Z^{-n}$$

$$= \sum_{n=0}^{\infty} Z^{-n} = \sum_{n=0}^{\infty} (Z^{-1})^n = \frac{1}{1-Z^{-1}}$$

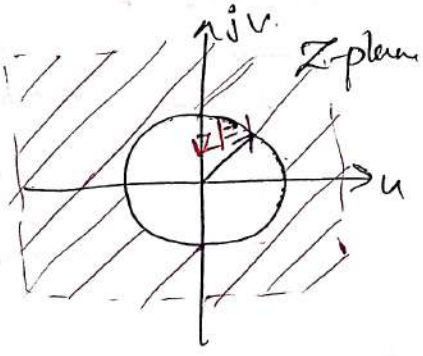
$$= \frac{1}{1-(1/2)} = \frac{1}{\frac{Z-1}{Z}} \Rightarrow \frac{Z}{Z-1}$$

Here the condition for convergence is, $0 < |Z^{-1}| < 1$.

$$\therefore |Z^{-1}| < 1 \Rightarrow \frac{1}{|Z|} < 1 \Rightarrow |Z| > 1$$

The term $|Z| = 1$ represents a circle of unit radius in Z-plane.

$\therefore$ the ROC is exterior of unit circle in Z-plane.

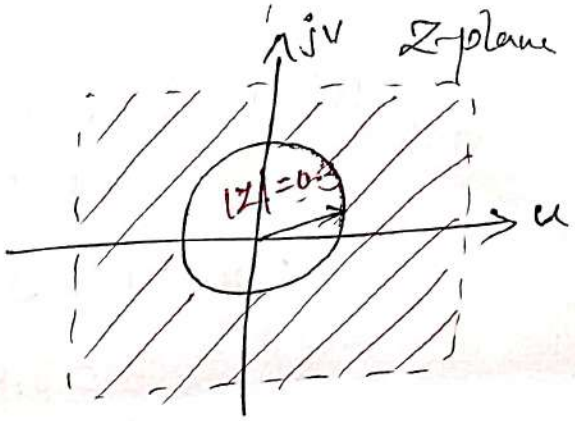


(b) $x(n) = 0.3^n u(n)$

The $u(n)$ is a discrete unit step sig. which is defined as.

$$u(n) = 1 \; ; \text{for } n \geq 0$$
$$= 0 \; ; \text{for } n < 0$$

$$\therefore x(n) = 0.3^n \; ; \text{for } n \geq 0$$
$$= 0 \; ; \text{for } n < 0$$



By the definition of Z-Transform

$$Z\{x(n)\} = X(Z) = \sum_{n=-\infty}^{\infty} x(n) \cdot Z^{-n} = \sum_{n=0}^{\infty} 0.3^n Z^{-n}$$

$$= \sum_{n=0}^{\infty} (0.3 Z^{-1})^n = \frac{1}{1 - 0.3 Z^{-1}}$$

$$\therefore X(Z) = \frac{1}{1 - 0.3(\frac{1}{Z})} = \frac{Z}{Z - 0.3}$$

Here the condition for convergence is, $0 < |0.3 Z^{-1}| < 1$.

$$\therefore |0.3 Z^{-1}| < 1 \Rightarrow \frac{0.3}{|Z|} < 1 \Rightarrow |Z| > 0.3$$

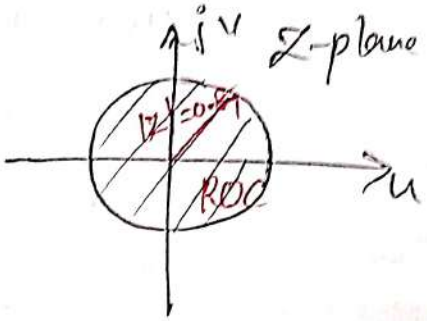
The term $|Z| = 0.3$ represents a circle of radius 0.3 in Z-plane. Therefore, the ROC is exterior of circle with radius of 0.3 in Z-plane.

(c) $x(n) = 0.8^{n} u(-n-1)$.

The $u(-n-1)$ is a discrete unit step sig, which is defined as

$$u(-n-1) = 0 \; ; \text{ for } n \geq 0$$
$$= 1 \; ; \text{ for } n \leq -1$$

$$\therefore x(n) = 0 \; ; \text{ for } n \geq 0$$
$$= 0.8^{n} \; ; \text{ for } n \leq -1$$



By the definition of Z-Transform,

$$Z\{x(n)\} = X(Z) = \sum_{n=-\infty}^{\infty} x(n) \cdot Z^{-n} = \sum_{n=-\infty}^{-1} (0.8)^{n} Z^{-n}$$

$$= \sum_{n=1}^{\infty} 0.8^{-n} Z^{+n} = \sum_{n=1}^{\infty} (0.8^{-1}Z)^{n} = \sum_{n=0}^{\infty} (0.8^{-1}Z)^{n} - 1$$

$$= \frac{1}{1-(0.8^{-1}Z)} - 1$$

$$= \frac{1}{1-\frac{Z}{0.8}} - 1 \Rightarrow \frac{0.8}{0.8-Z} - 1$$

$$\Rightarrow \frac{0.8 - 0.8 + Z}{0.8 - Z} = \frac{Z}{0.8-Z}$$

$$\Rightarrow \frac{-Z}{Z-0.8}$$

$(0.8^{-1}Z)^{0} = 1$

Using Infinite Geometric Series sum formula

Here the condition for convergence is, $0 < |0.8^{-1}Z| < 1$.

$$\therefore |0.8^{-1}Z| < 1 \Rightarrow \frac{|Z|}{0.8} < 1 \Rightarrow |Z| < 0.8$$

The term $|Z| = 0.8$, represents a circle of radius 0.8 in Z-plane. ∴ The ROC is interior of the circle of radius 0.8 in Z-plane.

(d) $x(n) = 0.3^n u(n) + 0.8^n u(-n-1)$.

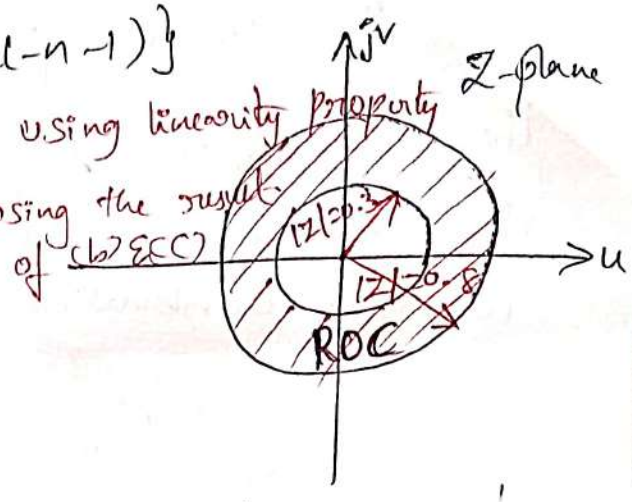
$$X(Z) = Z\{x(n)\} = Z\{0.3^n u(n) + 0.8^n u(-n-1)\}$$

$$= Z\{0.3^n u(n)\} + Z\{0.8^n u(-n-1)\}$$

using linearity property

$$= \frac{Z}{Z-0.3} - \frac{Z}{Z-0.8}$$

using the result of (b) ECC)



$$= \frac{Z(Z-0.8) - Z(Z-0.3)}{(Z-0.3)(Z-0.8)}$$

$$= \frac{Z^2 - 0.8Z - Z^2 + 0.3Z}{Z^2 - 0.8Z - 0.3Z + 0.24} \Rightarrow \frac{-0.5Z}{Z^2 - 1.1Z + 0.24} = X(Z)$$

Here the condition for convergence of $0.3^n u(n)$ is,

$$0 < |0.3Z^{-1}| < 1 \Rightarrow |Z| > 0.3$$

and the condition for convergence of $0.8^n u(-n-1)$ is

$$0 < |0.8^{-1} Z| < 1 \Rightarrow |Z| < 0.8$$

The term $|Z| = 0.8$, represent a circle of radius 0.8 in Z-plane and the terms $|Z| = 0.3$ represents a circle of radius 0.3 in Z-plane. Hence the common region of convergence for both the terms of $x(n)$ is the region in between the circles of radius $|Z| = 0.8$ and $|Z| = 0.3$ in the Z-plane.

Properties of Z-Transform:-

① **Linearity Property:** The linearity property of Z-Transform states that the Z-Transform of linear weighted combination of discrete time signals is equal to similar linear weighted combination of Z-Transform of individual discrete time sigs.

Let, $Z\{x_1(n)\} = X_1(Z)$ and $Z\{x_2(n)\} = X_2(Z)$ then by linearity property.

$$Z\{a_1x_1(n) + a_2x_2(n)\} = a_1X_1(Z) + a_2X_2(Z);$$

where a_1 & a_2 are constants.

Proof:- By defination of Z-Transform,

$$X_1(Z) = Z\{x_1(n)\} = \sum_{n=-\infty}^{\infty} x_1(n) \cdot z^{-n}$$

$$X_2(Z) = Z\{x_2(n)\} = \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}$$

$$\therefore Z\{a_1x_1(n) + a_2x_2(n)\} = \sum_{n=-\infty}^{\infty} [a_1x_1(n) + a_2x_2(n)] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} [a_1x_1(n)z^{-n} + a_2x_2(n) z^{-n}]$$

$$= \sum_{n=-\infty}^{\infty} a_1x_1(n) z^{-n} + \sum_{n=-\infty}^{\infty} a_2x_2(n) z^{-n}$$

$$= a_1 \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} + a_2 \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}.$$

$$= a_1X(Z) + a_2X_2(Z)$$

=

② Shifting Property:-

case(i): Two-Sided Z-Transform:-

The shifting property of Z-Transform states that, Z-Transform of a shifted sig shifted by m-units of time is obtained by multiplying z^m to Z-Transform of unshifted sig.

let $Z\{x(n)\} = X(z)$

Now, by shifting property,

$$Z\{x(n-m)\} = z^{-m} X(z)$$

$$Z\{x(n+m)\} = z^{m} X(z)$$

Proof:-

By defination of Z-Transform

$$X(z) = Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n-m) z^{-n}$$

let $n-m = p$;
$n = p+m$.
when $n \to -\infty$, $p \to -\infty$
when $n \to \infty$, $p \to \infty$

$$= \sum_{p=-\infty}^{\infty} x(p)\, z^{-(m+p)}$$

$$= \sum_{p=-\infty}^{\infty} x(p) \cdot z^{-m} \cdot z^{-p}$$

let, $p \to n$

$$= z^{-m} \cdot \sum_{p=-\infty}^{\infty} x(p) z^{-p}$$

$$= z^{-m} \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

$$= z^{-m} \cdot X(z).$$

By defination of Z-Transform

$$Z\{x(n+m)\} = \sum_{n=-\infty}^{\infty} x(n+m) z^{-m}$$

let $n+m = p$;
$\therefore n = p-m$
when $n \to -\infty$, $p \to -\infty$
when $n \to \infty$, $p \to \infty$

$$= \sum_{p=-\infty}^{\infty} x(p) \cdot z^{-(p-m)}$$

$$= \sum_{p=-\infty}^{\infty} x(p)\, z^{-p} \cdot z^{m}$$ let $p \to n$

$$= z^{m} \sum_{p=-\infty}^{\infty} x(p) z^{-p} = z^{m} \cdot \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}.$$

$$= z^{m} X(z).$$

Case (ii) : One-Sided Z-Transform :-

Let $x(n)$ be a discrete time sig defined in the range $0 < n < \infty$.

let, $Z\{x(n)\} = X(z)$

Now by shifting property,

$$Z\{x(n-m)\} = z^{-m} X(z) + \sum_{i=1}^{m} x(-i) z^{-(m-i)}$$

$$Z\{x(n+m)\} = z^{m} X(z) - \sum_{i=1}^{m-1} x(i) z^{(m-i)}$$

Proof :-

By defination of one-sided Z-Transform

$$X(z) = Z\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$\therefore Z\{x(n-m)\} = \sum_{n=0}^{\infty} x(n-m) z^{-n}$$ Multiply by z^{m} & z^{-m}

$$= \sum_{n=0}^{\infty} x(n-m) z^{-n} \cdot z^{m} z^{-m}$$

$$= z^{-m} \sum_{n=0}^{\infty} x(n-m) z^{-(n-m)}$$ let $n-m=p$

when $n \to 0$, $p \to -m$,

$$= z^{-m} \sum_{p=-m}^{\infty} x(p) \cdot z^{-p}$$

when $n \to \infty$, $p \to \infty$.

$$\Rightarrow z^{-m} \sum_{p=0}^{\infty} x(p) z^{-p} + z^{-m} \sum_{p=-m}^{-1} x(p) z^{-p}$$

$$= z^{-m} \sum_{p=0}^{\infty} x(p) \cdot z^{-p} + z^{-m} \sum_{p=1}^{m} x(-p) z^{p}$$

$$= z^{-m} \sum_{n=0}^{\infty} x(n) \cdot z^{-n} + z^{-m} \sum_{i=1}^{m} x(-i) z^{i}$$

Let $p=n$, 1st Summation

$$= z^{-m} X(z) + \sum_{i=1}^{m} x(-i) z^{-(m-i)} \longrightarrow (a)$$

Let $p=i$, 2nd Summ.

Note :-

In eqn (a) if $x(-i)$ for $i=1$ to m are zero then the shifting property of one-sided Z-Transform for delayed sig will be same as that for two-sided Z-Transform.

By defination of one-sided Z-Transform,

$$Z\{x(n+m)\} = \sum_{n=0}^{\infty} x(n+m) z^{-n}$$

$$= \sum_{n=0}^{\infty} x(n+m) z^{-n} z^{m} z^{-m}.$$

Multiply by z^{m} & z^{-m}

$$= z^{m} \sum_{n=0}^{\infty} x(n+m) z^{-(n+m)}$$

Let, $n+m=p$,

when $n \to 0$, $p \to m$,

when $n \to \infty$, $p \to \infty$.

$$= z^{m} \sum_{p=m}^{\infty} x(p) z^{-p}$$

$$= z^{m} \sum_{p=0}^{\infty} x(p) \cdot z^{-p} - z^{m} \sum_{p=0}^{m-1} x(p) z^{-p}.$$

$$= z^{m} \sum_{n=0}^{\infty} x(n) z^{-n} - z^{m} \sum_{i=0}^{m-1} x(i) z^{-i}$$

$$= z^{m} X(z) - \sum_{i=0}^{m-1} x(i) z^{m-i} \longrightarrow (b)$$

Let $p=n$, in 1st Summation let $p=i$, in 2nd summation,

Note :- In eqn (b) if $x(i)$ for $i=0$ to $m-1$ are zero then the shifting property of one sided Z-T. for advanced sig will be same as that for 2-sided Z-T.

(3) Multiplication by n (or Differentation in Z-domain):-

If $Z\{x(n)\} = X(z)$

then $Z\{n\,x(n)\} = -z \cdot \frac{d}{dz} X(z)$

In general, $Z\{n^m x(n)\} = \left(-z \cdot \frac{d}{dz}\right)^m X(z)$

$$\Rightarrow -z \cdot \frac{d}{dz}\left(-z \cdot \frac{d}{dz}\left(\cdots\left(-z \cdot \frac{d}{dz}\left(-z \frac{dx(z)}{dz}\right)\right)\cdots\right)\right)$$

(m-times.)

Proof:-

By defination of Z-Transform,

$$X(z) = Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}.$$

$$\therefore Z\{n\,x(n)\} = \sum_{n=-\infty}^{\infty} n\,x(n) z^{-n}.$$

$$= \sum_{n=-\infty}^{\infty} n x(n) \cdot z^{-n} z \cdot z^{-1}$$

Multiply by $z \& z^{-1}$

$$= -z \cdot \sum_{n=-\infty}^{\infty} x(n) \cdot \left[-n \cdot z^{-n-1}\right]$$

$$= -z \cdot \sum_{n=-\infty}^{\infty} x(n) \cdot \left[\frac{d}{dz} \cdot z^{-n}\right]$$

$\frac{d}{dz} z^{-n} = -n z^{-n-1}$

Interchanging Summation & differentiation

$$= -z \cdot \frac{d}{dz} \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

$$= -z \cdot \frac{d}{dz} X(z).$$

$$=$$

(4) Multiplication by an exponential sequence, a^n (or Scaling in Z-domain):-

If $Z\{x(n)\} = X(z)$

then $Z\{a^n x(n)\} = X(a^{-1}z)$

Proof:- By defination of Z-Transform

$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

$$\therefore Z\{a^n x(n)\} = \sum_{n=-\infty}^{\infty} a^n x(n) \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \cdot (a^{-1}z)^n$$

$$= X(a^{-1}z).$$

(5) Time reversal:-

If $Z\{x(n)\} = X(z)$

then $Z\{x(-n)\} = X(z^{-1})$

Proof:- By defination of Z-Transform.

$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$\therefore Z\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$$

let $p = -n$

when $n \to -\infty$, $p \to \infty$

when $n \to \infty$ $p \to -\infty$.

$$= \sum_{p=-\infty}^{\infty} x(p) z^{p}$$

$$= \sum_{p=-\infty}^{\infty} x(p) (z^{-1})^{-p} \Rightarrow X(z^{-1})$$

(6) Conjugation :-

If $Z\{x(n)\} = X(z)$

then $Z\{x^*(n)\} = X^*(z^*)$

Proof :- By defination of Z-Transform,

$$X(z) = Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

$$\therefore Z\{x^*(n)\} = \sum_{n=-\infty}^{\infty} x^*(n) z^{-n}$$

$$= \left[\sum_{n=-\infty}^{\infty} x(n) \cdot (z^*)^{-n}\right]^*$$

$$= \left[X(z^*)\right]^*$$

$$= X^*(z^*)$$

(7) Convolution Theorem :-

If $Z\{x_1(n)\} = X_1(z)$

and $Z\{x_2(n)\} = X_2(z)$

then $Z\{x_1(n) * x_2(n)\} = X_1(z) \cdot X_2(z)$

where, $x_1(n) * x_2(n) = \sum_{m=-\infty}^{\infty} x_1(m) x_2(n-m)$ ... ①

Proof :- By defination of Z-Transform

$$X_1(z) = Z\{x_1(n)\} = \sum_{n=-\infty}^{\infty} x_1(n) z^{-n}$$

$$X_2(z) = Z\{x_2(n)\} = \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}$$

$$\therefore Z\{x_1(n) * x_2(n)\} = \sum_{n=-\infty}^{\infty} \left[x_1(n) * x_2(n)\right] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{m=-\infty}^{\infty} x_1(m) * x_2(n-m) \right] z^{-n}$$ Using eqn ①

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x_1(m)\, x_2(n-m)\, z^{-n} z^{-m} z^{m}$$ Multiply by $z^{m} z^{-m}$

$$= \sum_{m=-\infty}^{\infty} x_1(m) z^{-m} \sum_{n=-\infty}^{\infty} x_2(n-m) z^{-(n-m)}$$ Let $n-m=p$

When $n \to -\infty$, $p \to -\infty$.

When $n \to \infty$, $p \to \infty$

$$= \sum_{m=-\infty}^{\infty} x_1(m) z^{-m} \cdot \sum_{p=-\infty}^{\infty} x_2(p) z^{-p}$$

Let $m=n$, 1st summation.

Let $p=n$, 2nd summation.

$$= \left[\sum_{n=-\infty}^{\infty} x_1(n) z^{-n} \right] \left[\sum_{n=-\infty}^{\infty} x_2(n) z^{-n} \right]$$

$$= X_1(z) \cdot X_2(z)$$

$$\Rightarrow$$

(8) Correlation Property:-

If $Z\{x(n)\} = X(z)$ and $Z\{y(n)\} = Y(z)$

then $Z\{r_{xy}(n)\} = X(z) \cdot Y(z^{-1})$

where, $r_{xy}(n) = \sum_{n=-\infty}^{\infty} x(n) \cdot y(n-m)$. $\longrightarrow$ (a)

Proof:- By definition of Z-Transform.

$$X(z) = Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$Y(z) = Z\{y(n)\} = \sum_{n=-\infty}^{\infty} y(n) z^{-n}.$$

$$\therefore Z\{r_{xy}(m)\} = \sum_{m=-\infty}^{\infty} r_{xy}(m)\, z^{-m}$$

$$= \sum_{m=-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} x(n).\, y(n-m)\right] z^{-m} \quad \rightarrow \text{using eqn (a)}$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(n).y(n-m)\, z^{-m} z^{-n} z^{n} \quad \text{Mul. by } z^{n} \& z^{-n}.$$

$$= \sum_{n=-\infty}^{\infty} x(n).\, z^{-n} \cdot \sum_{m=-\infty}^{\infty} y(n-m)\, z^{(n-m)}$$

$$= \sum_{n=-\infty}^{\infty} x(n).z^{-n} \cdot \sum_{p=-\infty}^{\infty} y(p).z^{p}. \qquad n-m=p,\ m=n-p$$

when, $m \rightarrow -\infty$, $p \rightarrow \infty$

when, $m \rightarrow \infty$, $p \rightarrow -\infty$

$$= \left[\sum_{n=-\infty}^{\infty} x(n) z^{-n}\right]\left[\sum_{p=-\infty}^{\infty} y(p).(z^{-1})^{-p}\right]$$

$$= X(z).Y(z^{-1}).$$

(9) Initial value Theorem:-

Let $x(n)$ be an one-sided sq defined in the range $0 \le n \le \infty$.

Now, if $Z\{x(n)\} = X(z)$.

then the initial value of $x(n)$ [i.e. $x(0)$] is given by.

$$x(0) = \lim_{z \to \infty} X(z).$$

Proof:- By defination of one sided - Z-Transform

$$X(z) = \sum_{n=0}^{\infty} x(n).z^{-n}$$

On expanding the above summation we get

$$X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \cdots$$

$$\therefore X(z) = x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \frac{x(3)}{z^3} + \cdots$$

On taking limit $z \to \infty$ in the above eqn we get

$$\lim_{z\to\infty} X(z) = \lim_{z\to\infty} \left[x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \frac{x(3)}{z^3} + \right]$$

$$= x(0) + 0 + 0 + 0 \cdots$$

$$\therefore x(0) = \lim_{z\to\infty} X(z)$$

(10) Final value Theorem:-

Let $x(n)$ be a one-sided sig defined in the range $0 \le n \le \infty$.

Now, if $Z\{x(n)\} = X(z)$

then the final value of $x(n)$ [i.e., $x(\infty)$] is given by,

$$x(\infty) = \lim_{z\to 1} (1 - z^{-1}) X(z) \quad \text{(or)} \quad x(\infty) = \lim_{z\to 1} \left(\frac{z-1}{z}\right) X(z).$$

Proof:- By defination of one sided Z-Transform

$$Z\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$\therefore Z\{x(n-1) - x(n)\} = \sum_{n=0}^{\infty} [x(n-1) - x(n)] z^{-n}$$

(LHS) (RHS)

$$LHS = Z\{x(n-1) - x(n)\}$$

$$= Z\{x(n-1)\} - Z\{x(n)\}$$ using linearity property

$= z^{-1} X(z) + x(-1) - X(z)$ — using shifting property

$= x(-1) - (1 - z^{-1}) X(z)$

$= \lim_{z \to 1} [x(-1) - (1 - z^{-1}) X(z)]$ — Taking limit $z \to 1$

$= x(-1) - \lim_{z \to 1} (1 - z^{-1}) X(z) \rightarrow (a)$

$RHS = \sum_{n=0}^{\infty} [x(n-1) - x(n)] z^{-n}$

$= \lim_{z \to 1} \sum_{n=0}^{\infty} [x(n-1) - x(n)] z^{-n}$ — "Taking limit $z \to 1$"

$= \sum_{n=0}^{\infty} [x(n-1) - x(n)]$ — on applying limit $z \to 1$, the term z^{-n} becomes unity.

$= \lim_{p \to \infty} \sum_{n=0}^{p} [x(n-1) - x(n)]$ — changing the summation index from 0 to p & then taking limit $p \to \infty$

$= \lim_{p \to \infty} \{ [x(-1) - x(0)] + [x(0) - x(1)] + [x(1) - x(2)] + \dots + [x(p-2) - x(p-1)] + [x(p-1) - x(p)] \}$

$= \lim_{p \to \infty} [x(-1) - x(p)] \Rightarrow x(-1) - x(\infty) \rightarrow (b)$

On equating eqn (a) & (b) we get,

$x(-1) - x(\infty) = x(-1) - \lim_{z \to 1} (1 - z^{-1}) \cdot X(z)$.

$\therefore x(\infty) = \lim_{z \to 1} (1 - z^{-1}) X(z)$

=

(12) Parseval's Relation:-

(11) Complex convolution Theorem (8) Multiplication in time domain:-

let, $Z\{x_1(n)\} = X_1(z)$ and $Z\{x_2(n)\} = X_2(z)$

Now, the complex convolution theorem states, that

$$Z\{x_1(n)\, x_2(n)\} = \frac{1}{2\pi j} \oint_C X_1(v) X_2\left(\frac{z}{v}\right) v^{-1} dv.$$

where v is dummy variable used for contour integration.

Proof :- let, $Z\{x_1(n)\} = X_1(z)$ and $Z\{x_2(n)\} = X_2(z)$

Now, by defination of inverse Z-transform.

$$x_1(n) = \frac{1}{2\pi j} \oint_C X_1(z) z^{n-1} dz$$

$$= \frac{1}{2\pi j} \oint_C X_1(v) v^{n-1} dv \quad \rightarrow (eq\ 1) \quad \{\text{let, } z = v.$$

Now, by defination of Z-Transform.

$$X_2(z) = \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}$$

Using the defination of Z-Transform, the $Z\{x_1(n) \cdot x_2(n)\}$ can be written as,

$$Z\{x_1(n) x_2(n)\} = \sum_{n=-\infty}^{\infty} x_1(n) \cdot x_2(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi j} \oint_C X_1(v) v^{n-1} dv\right] x_2(n) z^{-n}$$

(using eq 1)

$$= \frac{1}{2\pi j} \oint_C X_1(v) \sum_{n=-\infty}^{\infty} x_2(n) z^{-n} \cdot v^{n} v^{-1} dv$$

$$= \frac{1}{2\pi j} \oint_C X_1(v) \left[\sum_{n=-\infty}^{\infty} x_2(n) \left(\frac{z}{v}\right)^{-n}\right] v^{-1} dv$$

$$\Rightarrow \frac{1}{2\pi j} \oint_C X_1(v)\, X_2\left(\frac{z}{v}\right) \cdot v^{-1} dv.$$

(12) Parseval's Relation:-

If $Z\{x_1(n)\} = X_1(z)$ and $Z\{x_2(n)\} = X_2(z)$

Then the Parseval's relation states that

$$\sum_{n=-\infty}^{\infty} x_1(n) \cdot x_2^*(n) = \frac{1}{2\pi j} \oint_C X_1(z) \cdot X_2^*\left(\frac{1}{z^*}\right) z^{-1} dz.$$

Proof:- let, $Z\{x_1(n)\} = X_1(z)$ and

$$Z\{x_2(n)\} = X_2(z)$$

Now, by definition of Inverse Z-Transform.

$$x_1(n) = \frac{1}{2\pi j} \oint_C X_1(z) z^{n-1} dz$$

$$= \frac{1}{2\pi j} \oint_C X_1(v) \cdot v^{n-1} dv \rightarrow (a) \quad \text{let } z = v.$$

Now, by definition of Z-Transform

$$Z\{x_2(n)\} = \sum_{n=-\infty}^{\infty} x_2(n) \cdot z^{-n} \rightarrow (b)$$

Using the definition of Z-Transform, the $Z\{x_1(n) x_2^*(n)\}$ can be written as,

$$Z\{x_1(n) x_2^*(n)\} = \sum_{n=-\infty}^{\infty} x_1(n) x_2^*(n) z^{-n} \rightarrow (c)$$

On substituting for $x_1(n)$ from eqn (a) in eqn (c) we can write,

$$\sum_{n=-\infty}^{\infty} x_1(n) \cdot x_2^*(n) z^{-n} = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi j} \oint_C X_1(v) v^{n-1} dv \right] x_2^*(n) z^{-n}.$$

$$= \frac{1}{2\pi j} \oint_C X_1(v) \left[\sum_{n=-\infty}^{\infty} x_2^*(n) z^{-n} v^n \right] v^{-1} \, dv$$

$$= \frac{1}{2\pi j} \oint_C X_1(v) \left[\sum_{n=-\infty}^{\infty} x_2^*(n) \left(\frac{z}{v}\right)^n \right] v^{-1} \, dv.$$

$$= \frac{1}{2\pi j} \oint_C X_1(v) \left[\sum_{n=-\infty}^{\infty} x_2(n) \left(\frac{z^*}{v^*}\right)^{-n} \right] v^{-1} \, dv.$$

$$= \frac{1}{2\pi j} \oint_C X_1(v) \cdot X_2^*\left(\frac{z^*}{v^*}\right) v^{-1} \, dv.$$

Let us take limit $z \to 1$ in the above equation.

$$\therefore \lim_{z \to 1} \sum_{n=-\infty}^{\infty} x_1(n) x_2^*(n) z^{-n} = \lim_{z \to 1} \frac{1}{2\pi j} \oint_C X_1(v) X_2^*\left(\frac{z^*}{v^*}\right) v^{-1} dv$$

$$\sum_{n=-\infty}^{\infty} x_1(n) x_2^*(n) = \frac{1}{2\pi j} \oint_C X_1(v) X_2^*\left(\frac{1}{v^*}\right) v^{-1} dv.$$

$$\therefore \sum_{n=-\infty}^{\infty} x_1(n) x_2^*(n) = \frac{1}{2\pi j} \oint_C X_1(z) X_2^*\left(\frac{1}{z^*}\right) z^{-1} dz.$$

$\Rightarrow$

let $v = z$

Properties of ROC:-

1. The ROC is ring (or) disk in the Z-plane centered at the origin.
2. The ROC cannot contain any poles.
3. If $x(n)$ is a causal sequence then the ROC is entire Z-plane except at $Z=0$.
4. If $x(n)$ is non-causal sequence then the ROC is the entire Z-plane except at $Z=\infty$.
5. If $x(n)$ is a finite duration, two sided sequence the ROC is entire Z-plane except at $Z=0$ & $Z=\infty$.
6. If $x(n)$ is an infinite duration, two sided sequence, the ROC will consist of a ring in the Z-plane, bounded on the interior and exterior by a pole, not contain.
7. The ROC of a LTI stable system contains the unit circle.
8. The ROC must be a connected region.

Poles and Zeros in Z-plane:-

Poles and zeros in the Z-plane are representing roots of a system's transfer function.

Zeros occurs where the transfer function $H(Z)$ equals zeros, while poles occurs where

H(Z) approaches infinity.

Definations:-

Zeros are the roots of the numerator poly-nomial in the rational transfer function. $H(Z) = \frac{P(Z)}{Q(Z)}$, marked as a circles (○) on the Z-plane.

Poles are the roots of denominator polynomial marked as crosses (×), indicating points, where the system response becomes unbounded.

Z-plane Representation:-

The Z-plane is a complex plane with the real part on the horizontal axis and imaginary part on the vertical axis. where points are expressed as $Z = r \cdot e^{j\theta}$, poles inside the unit circle $|Z| < 1$ contribute to stable, decaying responses while those outside lead to instability.

Poles and Zeros and Z-plane:-

W.K.T. the Sln function $H(Z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$ → (1)

Zeros of the Sln function $H(Z)=0$

Poles " " " " $H(Z)=\infty$.

The Sln function contains both poles & zeros and hence the Sln is called pole-zero Sln. with N poles & M zeros.

In eqn (1) if $a_k=0$ fo $1 \le k \le N$, then

$$H(Z) = \sum_{k=0}^{M} b_k z^{-k} = \frac{1}{z^M} \sum_{k=0}^{M} b_k z^{M-k} \longrightarrow (2)$$

In this case $H(z)$ contain M zeros & an M^{th} order pole at origin $z=0$. Such a Sln is called an all zero Sln.

It has a finite duration impulse response (FIR), and it is called as FIR Sln (or) moving average (MA) Sln. On the other hand, if $b_k=0$ fo $1 \le k \le M$ the Sln function reduces to

$$H(Z) = \frac{b_0}{1 + \sum_{k=1}^{N} a_k z^{-k}} \longrightarrow (3)$$

In this case $H(z)$ contains N poles & N^{th} order zero at the origin $z=0$. Such Sln is called all pole Sln.

Due to the presence of poles, the impulse response of the Sm is infinite in duration, and hence it is an IIR systems.

The System Function:-

In general the Sm is described by a linear constant coefficient difference equation. of the Sm form.

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k) \rightarrow (1)$$

Taking Z-Transform on both sides, applying time shifting property, we obtain.

$$Y(z) = -\sum_{k=1}^{N} a_k Y(z) z^{-k} + \sum_{k=0}^{M} b_k X(z) z^{-k}$$

$$Y(z) + \sum_{k=1}^{N} a_k Y(z) z^{-k} = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$Y(z)\left[1 + \sum_{k=1}^{N} a_k z^{-k}\right] = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \rightarrow (2)$$

Eqn (2) is the transfer function / Sm fun. $Y(z)$ is the Z-Transform of the o/p sequence $y(n)$. and $X(z)$ is the Z-Transform of the i/p sequence $x(n)$.

If the i/p $x(n) = \delta(n)$, $X(Z)$ is equal to 1.

Then $Y(Z) = H(Z)$, where $H(Z)$ is the Z-Transform of the impulse response $h(n)$.

We can determine their corresponding Z-Transform $H(Z)$ and $X(Z)$, multiply them to obtain $Y(Z)$ and therefore determine $y(n)$ by evaluating the inverse Z-Transform of $Y(Z)$.

(1) Find the sbm function and impulse response of the sbm described by the difference equation $y(n) = \frac{1}{5} y(n-1) + x(n)$.

Sol:- Given $y(n) = \frac{1}{5} y(n-1) + x(n)$

Talking Z-Transform on both sides we get

$$Y(Z) = \frac{1}{5} Z^{-1} Y(Z) + X(Z)$$

(a)

$$Y(Z) - \frac{1}{5} Z^{-1} Y(Z) = X(Z)$$

$$Y(Z)\left[1 - \frac{1}{5} Z^{-1}\right] = X(Z)$$

Now, the sbm function is $H(Z) = \frac{Y(Z)}{X(Z)} = \frac{1}{1 - \frac{1}{5} Z^{-1}}$

The inverse Z-Transform. $h(n) = \left(\frac{1}{5}\right)^n u(n)$

(2) Find the Sln function and the impulse response of the system described by the difference equation $y(n) = x(n) + 2x(n-1) - 4x(n-2) + x(n-3)$.

Sol:- Given $y(n) = x(n) + 2x(n-1) - 4x(n-2) + x(n-3)$

Taking Z-Transform on both sides.

$$Y(z) = X(z) + 2z^{-1}X(z) - 4z^{-2}X(z) + z^{-3}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = 1 + 2z^{-1} - 4z^{-2} + z^{-3}$$

$$= h(0)z^{0} + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3}$$

from which we can find

$h(0) = 1$; $h(1) = 2$, $h(2) = -4$, $h(3) = 1$

$$\therefore h(n) = \{1, 2, -4, 1\}$$

(3) Find the Sln function and impulse response of the Sln described by the difference equation.

(a) $y(n) = y(n-1) + 2y(n-2) + x(n)$

Ans: $\dfrac{1}{1 - z^{-1} + 2z^{-2}}$

(b) $y(n) = 2x(n-1) + \frac{1}{2}x(n-2) + x(n-3) + \frac{1}{3}x(n-4)$

Ans:- $2z^{-1} + \frac{1}{2}z^{-2} + z^{-3} + \frac{1}{3}z^{-4}$

* Determine the pole-zero plot for the slm described by difference equation $y(n) - \frac{3}{4} y(n-1) + \frac{1}{8} y(n-2) = x(n) - x(n-1)$.

Sol:- Given the difference equation

$$y(n) - \frac{3}{4} y(n-1) + \frac{1}{8} y(n-2) = x(n) - x(n-1)$$

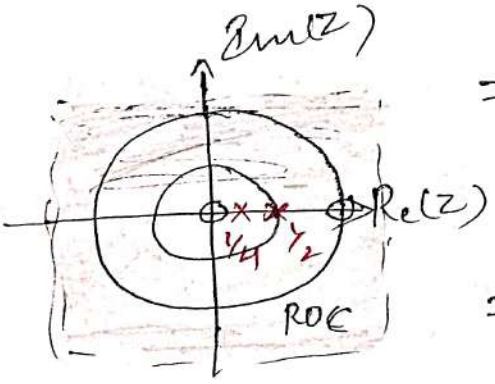
Taking Z-Transform on both sides, we obtain

$$Y(z) - \frac{3}{4} z^{-1} Y(z) + \frac{1}{8} z^{-2} Y(z) = X(z) - z^{-1} X(z)$$

$$Y(z)\left[1 - \frac{3}{4} z^{-1} + \frac{1}{8} z^{-2}\right] = X(z)\left[1 - z^{-1}\right]$$

The system function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 - \frac{3}{4} z^{-1} + \frac{1}{8} z^{-2}}$$



$$= \frac{1 - z^{-1}}{\left(1 - \frac{1}{4} z^{-1}\right)\left(1 - \frac{1}{2} z^{-1}\right)} \quad \therefore \text{ROC } |z| > \frac{1}{2}$$

$$= \frac{z(z-1)}{\left(z - \frac{1}{4}\right)\left(z - \frac{1}{2}\right)}$$

Zeros at $z = 0, 1$ & Poles at $z = \frac{1}{4}, \frac{1}{2}$.

From the fig. we can observe the

① ROC of the slm function includes unit circle.

② ROC of the slm function cannot contain any poles.

(2) Find the pole-zero plot for the s/m described by difference equation. $y(n) = \frac{5}{6} y(n-1) - \frac{1}{6} y(n-2) + x(n) - x(n-1)$

Sol:- Ans : Zeros $z = 0, 1$

Poles $= \frac{1}{2}, \frac{1}{3}$.

Stability Criterion (BIBO stability):-

The s/m function is stable if and only if all poles of $H(z)$ are inside the unit circle.

The necessary and sufficient condition for a causal linear time-invariant system to be BIBO stable is ~~for~~ a ~~causal system~~.

~~$H(z) = \sum_{n=0}^{\infty} h(n) < \infty$~~.

for a causal system

$$H(z) = \sum_{n=0}^{\infty} h(n) \cdot z^{-n}$$

$$|H(z)| = \sum_{n=0}^{\infty} |h(n) z^{-n}| \leq \sum_{n=0}^{\infty} |h(n)| |z^{-n}|.$$

when evaluated on the unit circle (i.e) $|z| = 1$)

$$|H(z)| \leq \sum_{n=0}^{\infty} |h(n)| < \infty$$

The Slm is stable the ROC of the Slm function includes the unit circle. But for a causal slm the ROC is exterior of some circle of radius r.

A causal & stable slm must have a slm function that converges for $|z| > r < 1$. Since ROC can't contain any poles of $H(z)$.

A causal linear time invariant Slm is BIBO stable if and only if all poles of $H(z)$ are inside the unit circle.

* Discuss on stability of the slm for given differential equation.

$$y(n) - \frac{3}{4} y(n-1) + \frac{1}{8} y(n-2) = x(n) + x(n-1).$$

Sol:- The pole zero plot of the given slm is poles at $z_1 = \frac{1}{4}$, & $z_2 = \frac{1}{2}$

Both poles are inside the unit circle, so the system is stable.

Relationship between the Fourier Transform and Z-Transform:

The Relationship btw transfer function $H(e^{j\omega})$ and the Sln function $H(z)$ of a Sln. This relationship is evaluating the Fourier transform in the Z-plane. We know the Z-Transform of $h(n)$ is given by.

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n} \longrightarrow (1)$$

where $z = e^{j\omega}$. $\longrightarrow (2)$

Substitute $z = re^{j\omega}$ in eqn (1). we obtain

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n) (re^{j\omega})^{-n} \longrightarrow (3)$$

The F.T. of $h(n)$ is given by

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n) e^{j\omega n} \longrightarrow (4)$$

eqn (3) & (4) are identical when $r=1$. In the Z-plane this corresponds to the locus of point on the unit circle $|z|=1$. Hence $H(e^{j\omega})$ is equal to $H(z)$ evaluated along the unit circle. (or)

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}.$$

For $H(e^{j\omega})$ to exist, the ROC of $H(z)$ must include the unit circle.

* Evaluate the frequency response of the Sln described by the Sln function $H(z) = \frac{1}{1-0.5z^{-1}}$

Sol:- Given $H(z) = \frac{1}{1-0.5z^{-1}} = \frac{z}{z-0.5}$ ROC; $|z| > \frac{1}{2}$

Since ROC includes unit circle $H(e^{j\omega})$ exists.

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}} = \frac{1}{1-0.5e^{-j\omega}} = \frac{e^{j\omega}}{e^{j\omega}-0.5}$$

$$\Rightarrow \frac{\cos\omega + j\sin\omega}{\cos\omega - 0.5 + j\sin\omega}$$

$\left\{ e^{j\omega} = \cos\omega + j\sin\omega \right.$

The magnitude response is

$$|H(e^{j\omega})| = \frac{|\cos\omega + j\sin\omega|}{|\cos\omega + j\sin\omega - 0.5|} = \frac{1}{\sqrt{(\cos\omega - 0.5)^2 + (\sin^2\omega)^2}}$$

$$\Rightarrow \frac{1}{\sqrt{1.25 - \cos\omega}}$$

and the phase response is $\angle H(e^{j\omega}) = \omega - \tan^{-1}\frac{\sin\omega}{\cos\omega - 0.5}$

(b) Evaluate the frequency response of the Sln described by the Sln function.

Sol:- $H(z) = \frac{1}{1-3z^{-1}}$

Ans:- $H(e^{j\omega})$ does not exist.

Relationship between S-plane and Z-plane:-

The laplace transform of a continuous causal sig $x(t)$ is defined as

$$X(s) = \int_0^\infty x(t) e^{-st} dt \longrightarrow (1)$$

If we sample the sig with a sampling time T we get a discrete sig $x^*(t)$ where

$$x^*(t) = \sum_{n=0}^{\infty} x(nT) \delta(t-nT) \longrightarrow (2)$$

The laplace transform of $x^*(t)$ can be written as

$$X^*(s) = \sum_{n=0}^{\infty} x(nT) e^{-nTs} \longrightarrow (3)$$

Substitute $Z = e^{sT}$ in eqn (3) we get

$$Z\text{-Transform of } x^*(t) = X(Z) = \sum_{n=0}^{\infty} x(nT) Z^{-n} \longrightarrow (4)$$

Therefore the relation between s-plane and Z-plane can be described by the eqn $Z = e^{sT} \longrightarrow (5)$

Let $Z = re^{j\omega}$ and $s = \sigma + j\Omega \longrightarrow (6)$

Substituting these values in eqn (5) we obtain

$$Z = re^{j\omega} = e^{(\sigma + j\Omega)T} = e^{\sigma T} \cdot e^{j\Omega T} \longrightarrow (7)$$

From eqn (7) we find $|Z| = e^{\sigma T}$ and $\omega = \Omega T$, $\sigma = 0$ gives $|Z| = 1$; $\sigma < 0$ gives $|Z| < 1$

and $\sigma > 0$ gives $|z| \geq 1$.

Since $\sigma = 0$ gives $|z| = 1$, the $j\Omega$ axis of s-plane maps into the unit circle.

The left half of s-plane where $\sigma < 0$ maps into the inside the unit circle ($|z| < 1$).

The Right half of s-plane RHP where $\sigma \geq 0$ maps into the outside of the unit circle ($|z| > 1$). The mapping is as shown in fig.

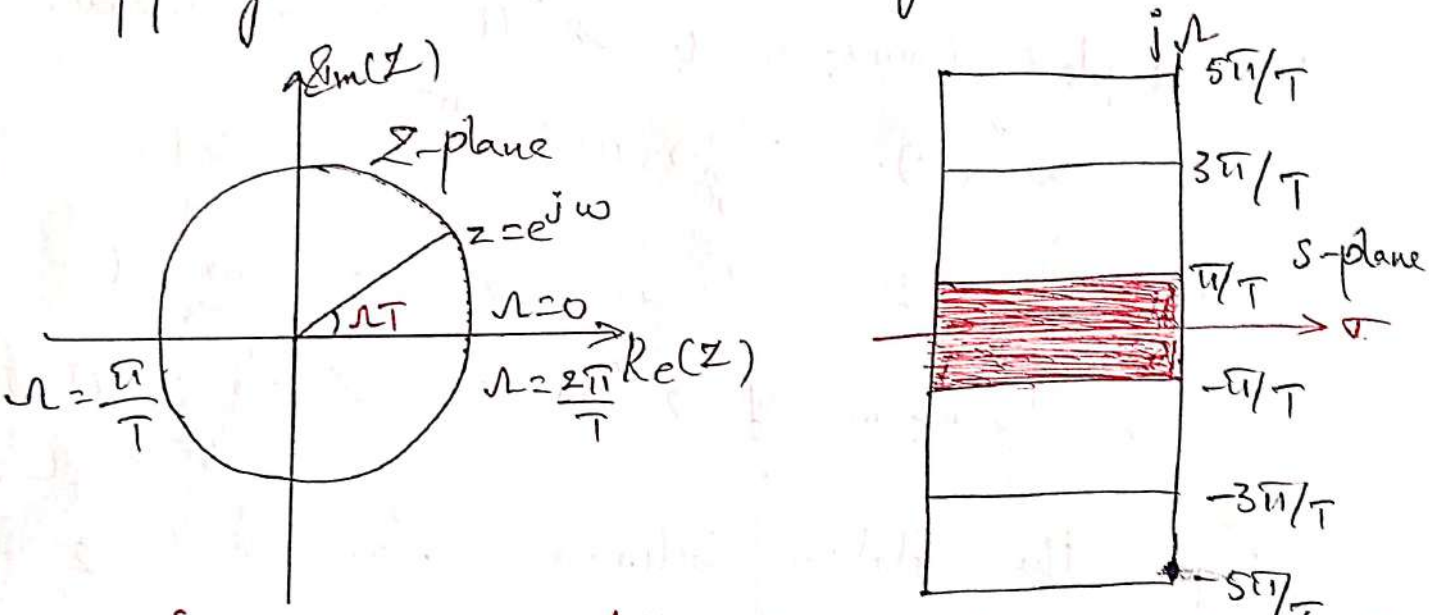


fig: Mapping from the s-plane to the z-plane.

From the fig a single horizontal strip of width $\frac{2\pi}{T}$, from $\Omega = -\pi/T$ to $\Omega = \frac{\pi}{T}$ completely maps into the inside the unit circle and the horizontal strips between π/T and $3\pi/T$, b/w $-\pi/T$ and $-\frac{3\pi}{T}$.

[In general b/w $(2n-1)\pi/T$ and $(2n+1)\pi/T$ where $n = 0, \pm 1, \pm 2, \ldots$) are mapped.

again into the inside the unit circle.

Thus many points in the s-plane are mapped to a single point in the z-plane, causing aliasing effect. That is when we sample two sinusoidal sigs of the frequencies which differ by a multiple of the sampling frequency, we can't distinguish between the result.

Inverse Z-Transform:-

There are four methods that are used for evaluation of the inverse Z-Transform.

1. Long division method
2. Partial fraction expansion method
3. Residue method
4. Convolution method.

Long Division Method:-

From the one-sided Z-Transform eqn

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

can be expanded into an infinite powers of z^{-1} as follows.

$$X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3) + \cdots + x(k)z^{-k} + \cdots$$
$$x(n)z^{-n} + \cdots \longrightarrow (1)$$

Thus the value of $x(n)$ at any instant of time are the coefficients of z^{-k}. If $X(z)$ is given as a ratio of two polynomials, the coefficients $x(0), x(1), \ldots, x(n)$ can be obtained by synthetic division of the numerator by the denominator as follows.

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

$$= x(0) + x(1) z^{-1} + \cdots \longrightarrow (2)$$

∴ The values of the sequence that represent the inverse Z-Transform are $x(n)$ for $n \geq 0$. which is a causal sequence. The Z-Transform $X(z)$ can also be represented as

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + b_N z^{-N}}$$

$$= \cdots + x(-n) z^{n} + \cdots + x(-k) z^{k} + \cdots$$

$$+ x(-2) z^{2} + x(-1) z + x(0) \longrightarrow (3)$$

∴ The values of the sequence that represent the inverse Z-Transform are $x(n)$ for $n \leq 0$ which is a non-causal sequence

=

* Find the inverse Z-Transform of $X(z) = \frac{z+0.2}{(z+0.5)(z-1}$

$|z|>1$ and (b) Find the B-Z.T. of

$$X(z) = \frac{z}{(z-3)(z-4)}, \quad |z| <$$

Sol:- (a) From the ROC, we find that $x(n)$ is a causal sequence., given $X(z) = \frac{z+0.2}{z^2-0.5z-0.5}$

Dividing, the numerator by denominator.

$$z^2 - 0.5z - 0.5 \;\Big|\; z^{-1} + 0.7z^{-2} + 0.85z^{-3} + 0.7752z^{-4}$$

$$z + 0.2$$
$$z - 0.5 - 0.5z^{-1}$$
$$0.7 + 0.5z^{-1}$$
$$0.7 - 0.35z^{-1} - 0.35z^{-2}$$
$$0.85z^{-1} + 0.35z^{-2}$$
$$0.85z^{-1} - 0.425z^{-2} - 0.425z^{-3}$$
$$0.7752z^{-2} + 0.4252z^{-3}$$
$$0.7752z^{-2} - 0.3876z^{-3} - 0.3876z^{-4}$$

$$\therefore X(z) = z^{-1} + 0.7z^{-2} + 0.85z^{-3} + 0.7752z^{-4} + \cdots$$

$$= \sum_{n=0}^{\infty} x(n) z^{-n}.$$

where $x(0)=0$, $x(1)=1$, $x(2)=0.7$, $x(3)=0.85$,

$x(4)=0.775$ so on.

(16) From the ROC, we know $x(n)$ is a anticausal sequence.

$$X(z) = \frac{z}{(z-3)(z-4)} = \frac{z}{z^2 - 7z + 12}$$

dividing numerator by denominator

$$12 - 7z + z^2 \;\Big|\; z + \frac{7}{12}z^2 + \frac{37}{144}z^3$$

$$z - \frac{7}{12}z^2 + \frac{1}{12}z^3$$

$$\frac{7}{12}z^2 - \frac{1}{12}z^3$$

$$\frac{7}{12}z^2 - \frac{49}{144}z^3 + \frac{7}{144}z^4$$

$$\frac{37}{144}z^3 - \frac{7}{144}z^4$$

$$\frac{37}{144}z^3 - \frac{259}{1728}z^4 + \frac{37}{1728}z^5$$

$$X(z) = \frac{1}{12}z + \frac{7}{144}z^2 + \frac{37}{1728}z^3 + \cdots$$

$$= \sum_{n=-\infty}^{-1} x(n) z^{-n}$$

$\therefore x(0) = 0,\ x(-1) = \frac{1}{12},\ x(-2) = \frac{7}{144},\ x(-3) = \frac{37}{1728}$

So on

=

Inverse Z-Transform by Power Series Expansion Method:-

Let $X(z)$ be Z-Transform of $x(n)$, and $X(z)$ is a rational function of z as

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

On dividing the numerator polynomial $N(z)$ by denominator polynominator $D(z)$ we can express $x(z)$ as a power series of z. It is possible to express $x(z)$ as positive power of z (or) as negative power of z (or) with both +ve & -ve power of z as shown below.

Case (i): $X(z) = \frac{N(z)}{D(z)} = c_0 + c_1 z^{-1} + c_2 z^{-2} + \cdots$

Case (ii): $X(z) = \frac{N(z)}{D(z)} = d_0 + d_1 z + d_2 z^2 + \cdots$

Case (iii): $X(z) = \frac{N(z)}{D(z)} = \cdots + e_{-2} z^2 + e_{-1} z^1 + e_0 + e_1 z^{-1} + e_2 z^{-2} + \cdots$

The case i - power series of z is obtained when the ROC is exterior of a circle of radius r in Z-plane (i.e ROC is $|z| > 1$).

Case-ii power series of z obtained when the ROC is interior of a circle of radius 'r' in z-plane (i.e ROC is $|z|<1$)

In case-iii power series of z obtained the ROC in btw two circles of radius r_1 & r_2 in z-plane (i.e. ROC is $r_1<|z|<r_2$).

There the 3 different methods that are consider in Inverse Z-Transform

1. Residue method
2. Partial fraction Expansion method
3. Long division / Power series expansion method

* Determine the Inverse Z-Transform of the function, $X(z)=\frac{3+2z^{-1}+z^{-2}}{1-3z^{-1}+2z^{-2}}$ by the above 3 following methods.

Residue Method:-

Given that, $X(z)=\frac{3+2z^{-1}+z^{-2}}{1-3z^{-1}+2z^{-2}}$

$$=\frac{z^{-2}[3z^2+2z+1]}{z^{-2}[z^2-3z+2]}=\frac{3z^2+2z+1}{z^2-3z+2}.$$

Divide the numerator polynomial by denominator polynomial and express $X(z)$ as

$$X(z) = \frac{3z^2+2z+1}{z^2-3z+2} = 3 + \frac{11z-5}{z^2-3z+2}$$

$$\Rightarrow 3 + \frac{11z-5}{(z-1)(z-2)}$$

$z^2-3z+2 \;\big|\; 3z^2+2z+1$

$3z^2-9z+6$

$+11z-5$

Let $X_1(z) = 3$, $X_2(z) = \frac{11z-5}{z^2-3z+2}$

$\therefore X(z) = X_1(z) + X_2(z)$

$$x(n) = z^{-1}\{X(z)\} = z^{-1}\{X_1(z)\} + z^{-1}\{X_2(z)\}$$

$$= z^{-1}\{3\} + z^{-1}\{X_2(z)\}$$

$$= 3\delta(n) + \sum_{i=1}^{N}\left[(z-p_i)\,X_2(z)\,z^{n-1}\Big|_{z=p_i}\right]$$

$\Rightarrow 3\delta(n) + (z-1)\frac{11z-5}{(z-1)(z-2)}z^{n-1}\Big|_{z=1} +$ using residue Theorem

$$(z-2)\frac{11z-5}{(z-1)(z-2)}z^{n-1}\Big|_{z=2}$$

$$\Rightarrow 3\delta(n) + \frac{11\cdot-5}{1-2}(1)^{n-1} + \frac{11\times2-5}{2-1}2^{n-1}$$

$$\Rightarrow 3\delta(n) + \frac{6}{-1}(1)^{n-1} + \frac{17}{1}2^{n-1}$$

$$\Rightarrow x(n) = 3\delta(n) + 6\,u(n-1) + 17\,2^{n-1}u(n-1)$$

$$\Rightarrow 3\delta(n) + \left[6 + 17\,2^{n-1}\right]u(n-1)$$

when $n=0$, $x(0) = 3 - 0 + 0 = 3$

$n=1$, $x(1) = 0 + 6 + 17 \times 2^0 = 23$

$n=2$, $x(2) = 0 + 6 + 17 \times 2^1 = 40$

$\therefore x(n) = \{3, 23, 40 - -\}$

↑

Method 2 : Partial Fraction Expansion Method.

Given that, $X(z) = \dfrac{3 + 2z^{-1} + z^{-2}}{1 - 3z^{-1} + 2z^{-2}} \Rightarrow \dfrac{3z^2 + 2z + 1}{(z-1)(z-2)}$

$\therefore \dfrac{X(z)}{z} = \dfrac{3z^2 + 2z + 1}{z(z-1)(z-2)}$

Let, $\dfrac{X(z)}{z} = \dfrac{3z^2 + 2z + 1}{z(z-1)(z-2)} = \dfrac{A_1}{z} + \dfrac{A_2}{z-1} + \dfrac{A_3}{z-2}$

Now, $A_1 = z \cdot \dfrac{X(z)}{z}\Big|_{z=0}$

$= z \cdot \dfrac{3z^2 + 2z + 1}{z(z-1)(z-2)}\Big|_{z=0} \Rightarrow \dfrac{0 + 0 + 1}{(0-1)(0-2)} = 0.5$

$A_2 = (z-1) \dfrac{3z^2 + 2z + 1}{z(z-1)(z-2)}\Big|_{z=1} \Rightarrow \dfrac{3 + 2 + 1}{1 \times (1-2)} = -6$

$A_3 = (z-2) \dfrac{3z^2 + 2z + 1}{z(z-1)(z-2)}\Big|_{z=2} \Rightarrow \dfrac{3(4) + 2(2) + 1}{2(2-1)}$

$\Rightarrow \dfrac{12 + 4 + 1}{2} = \dfrac{17}{2} = 8.5.$

$$\frac{X(z)}{z} = \frac{0.5}{z} - \frac{6}{z-1} + \frac{8.5}{z-2}$$

On taking inverse Z-Transform of $X(z)$ we get

$$x(n) = 0.5\delta(n) - 6u(n) + 8.5(2)^n u(n)$$

$$= 0.5\delta(n) + [-6 + 8.5(2)^n]u(n)$$

$Z\{\delta(n)\} = 1$

$Z\{u(n)\} = \frac{z}{z-1}$

$Z\{a^n u(n)\} = \frac{z}{z-a}$

where $n=0$, $x(0) = 0.5 - 6 + 8.5 \times 2^0 = 3$

$n=1$, $x(1) = 0 - 6 + 8.5 \times 2^1 = 11$

$n=2$, $x(2) = 0 - 6 + 8.5 \times 2^2 = 28$

$n=3$, $x(3) = 0 - 6 + 8.5 \times 2^3 = 62$

$n=4$, $x(4) = 0 - 6 + 8.5 \times 2^4 = 130$

$\therefore x(n) = \{3, 11, 28, 62, 130 \ldots\}$

↑

Method 3: Power Series Expansion Method:

Given that $X(z) = \frac{3 + 2z^{-1} + z^{-2}}{1 - 3z^{-1} + 2z^{-2}}$

Let us devide the numerator polynomial by denominator polynomial

$1 - 3z^{-1} + 2z^{-2}$

as

$$1-3z^{-1}+2z^{-2} \;\Big|\; 3+11z^{-1}+28z^{-2}+62z^{-3}+130z^{-4}+\cdots$$

$3+2z^{-1}+z^{-2}$

$3-9z^{-1}+6z^{-2}$

$11z^{-1}-5z^{-2}$

$11z^{-1}-33z^{-2}+22z^{-3}$

$28z^{-2}-22z^{-3}$

$28z^{-2}-84z^{-3}+56z^{-4}$

$62z^{-3}-56z^{-4}$

$62z^{-3}-186z^{-4}+124z^{-5}$

$130z^{-4}+124z^{-5}$

$\vdots$

$$\therefore X(z)=\frac{3+2z^{-1}+z^{-2}}{1-3z^{-1}+2z^{-2}}=3+11z^{-1}+28z^{-2}+62z^{-3}+130z^{-4}+ \longrightarrow (1)$$

Let, $x(n)$ be inverse Z-transform of $X(z)$

Now, by defination of Z-Transform,

$$X(z)=\sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$=\cdots+x(0)+x(1)z^{-1}+x(2)z^{-2}+x(3)z^{-3}+x(4)z^{-4}+\cdots \longrightarrow (2)$$

On comparing eqn (1) & (2)

$x(0)=3$, $x(1)=11$, $x(2)=28$, $x(3)=62$, $x(4)=130$ and so on

$\therefore x(n)=\{3, 11, 28, 62, 130, \cdots\}$

* Determine the inverse Z-Transform of the following Z-domain functions.

(a) $X(z) = \dfrac{3z^2+2z+1}{z^2+4z+3}$

Sol:- On Dividing the numerator by denominator, the X(z) can be expressed as

$$z^2+4z+3 \;\overline{)\; 3z^2+2z+1 \;\;(3}$$
$$3z^2+12z+9$$
$$-10z-8$$

$$X(z) = \frac{3z^2+2z+1}{z^2+4z+3} \Rightarrow 3 + \frac{-10z-8}{z^2+4z+3}$$

$$\Rightarrow 3 + \frac{-10z-8}{(z+1)(z+3)}$$

By partial fraction Expansion we get

$$X(z) = 3 + \frac{A_1}{z+1} + \frac{A_2}{z+3}$$

$$A_1 = (z+1)\left.\frac{-10z-8}{(z+1)(z+3)}\right|_{z=-1} \Rightarrow \frac{-10(-1)-8}{-1+3}$$

$$\Rightarrow \frac{10-8}{2} = \frac{2}{2} = 1$$

$$A_2 = (z+3)\left.\frac{-10z-8}{(z+1)(z+3)}\right|_{z=-3} \Rightarrow \frac{-10(-3)-8}{-3+3} = \frac{30-8}{-2}$$

$$= \frac{22}{-2} = -11$$

$$\therefore X(z) = 3 + \frac{1}{z+1} - \frac{11}{z+3}$$ {"multiply & divide by z"}

$$\Rightarrow 3 + \frac{1}{z} \cdot \frac{z}{z-(-1)} - 11\frac{1}{z}\cdot\frac{z}{z-(-3)}$$

$$\Rightarrow 3 + z^{-1}\cdot\frac{z}{z-(-1)} - 11z^{-1}\frac{z}{z-(-3)}$$

On taking inverse Z-Transform of $X(z)$, we get,

$$x(n) = 3\delta(n) + (-1)^{n-1}u(n-1) - 4(-3)^{n-1}u(n-1)$$

$$= 3\delta(n) + \left[(-1)^{n-1} - 4(-3)^{n-1}\right]u(n-1)$$

when $n=0$, $x(0) = 3+0+0 = 3$

$n=1$, $x(1) = 0+1-11 = -10$

$n=2$, $x(2) = 0-1+33 = 32$

$n=3$, $x(3) = 0+1-99 = -98$

$n=4$, $x(4) = 0-1+297 = 296$

$$\therefore x(n) = \{3, -10, 32, -98, 296, \ldots\}$$

$\uparrow$

$Z\{\delta(n)\} = 1$

$Z\{a^n u(n)\} = \frac{z}{z-a}$

If $Z\{a^n u(n)\} = \frac{z}{z-a}$ then by time shifting property

$Z\{a^{(n-1)}u(n-1)\} = z^{-1}\frac{z}{z-a}$

Alternate Method:-

$$X(z) = \frac{3z^2+2z+1}{z^2+4z+3} \Rightarrow \frac{X(z)}{z} = \frac{3z^2+2z+1}{z(z^2+4z+3)}$$

$$\Rightarrow \frac{3z^2+2z+1}{z(z+1)(z+3)}.$$

By partial fraction expansion technique $\frac{X(z)}{z}$ can be expressed as,

$$\frac{X(z)}{z} = \frac{3z^2+2z+1}{z(z+1)(z+3)} \Rightarrow \frac{A_1}{z} + \frac{A_2}{z+1} + \frac{A_3}{z+3}.$$

$$A_1 = z \cdot \frac{X(z)}{z}\Big|_{z=0} \Rightarrow z \cdot \frac{3z^2+2z+1}{z(z+1)(z+2)}\Big|_{z=0} = \frac{0+0+1}{(0+1)(0+3)} = \frac{1}{3}$$

$$A_2 = (z+1)\left.\frac{X(z)}{z}\right|_{z=-1} \Rightarrow (z+1)\frac{3z^2+2z+1}{z(z+1)(z+3)}$$

$$\Rightarrow \frac{3(-1)^2+2(-1)+1}{-1\times(-1+3)} \Rightarrow \frac{3-2+1}{-2} = \frac{2}{-2} = -$$

$$A_3 = (z+3)\left.\frac{X(z)}{z}\right|_{z=-3} \Rightarrow (z+3)\frac{3z^2+2z+1}{z(z+1)(z+3)}$$

$$\Rightarrow \frac{3(-3)^2+2(-3)+1}{-3(-3+1)} = \frac{27-6+1}{6}$$

$$= \frac{22}{6} = 11/3$$

$$\therefore \frac{X(z)}{z} = \frac{1}{3}\cdot\frac{1}{z} - \frac{1}{z+1} + \frac{11}{3}\cdot\frac{1}{z+3}$$

$$\therefore X(z) = \frac{1}{3}\cdot\frac{z}{z} - \frac{z}{z+1} + \frac{11}{3}\cdot\frac{z}{z+3}$$

$$X(z) = 1/3 - \frac{z}{z-(-1)} + 11/3\cdot\frac{z}{z-(-3)}$$

$Z\{\delta(n)\} = 1$

$Z\{a^n u(n)\} = \frac{z}{z-a}$

On taking inverse Z-Transform of $X(z)$, we get

$$x(n) = 1/3\,\delta(n) - (-1)^n u(n) + 11/3\,(-3)^n u(n)$$

$$= \frac{1}{3}\delta(n) + \left[-(-1)^n + 11/3\,(-3)^n\right]u(n).$$

when $n=0$, $x(0) = 1/3 - 1 + \frac{11}{3} = 3$

$n=1$, $x(1) = 0+1+11/3\,(-3) = -10$

$n=2$, $x(2) = 0-1+11/3\,(-3)^2 = 32$

$n=3$, $x(3) = 0+1+11/3\,(-3)^3 = -98$

$$\therefore x(n) = \{\underset{\uparrow}{3}, -10, 32, -98, 296 \ldots\}$$

Analysis of LTI Discrete time system using Z-Transform:-

Transfer function of LTI Discrete Time Slm:-

Let $x(n)$ be the input and $y(n)$ be the o/p of an LTI discrete time slm. The equation to the i/p - o/p relation of an LTI discrete time slm is

$$y(n) = -\sum_{m=1}^{N} a_m y(n-m) + \sum_{m=0}^{M} b_m x(n-m) \quad \longrightarrow (1)$$

The above eqn. is a constant coefficient difference eqn. and N is the order of the slm.

let us take the Z-Transform with both sides of eqn with zero initial condition (i.e $y(n)=0$ for $n<0$ and $x(n)=0$ for $n<0$).

$$\therefore Z\{y(n)\} = Z\left\{-\sum_{m=1}^{N} a_m y(n-m) + \sum_{m=0}^{M} b_m x(n-m)\right\}$$

$$= Z\left\{-\sum_{m=1}^{N} a_m y(n-m)\right\} + Z\left\{\sum_{m=0}^{M} b_m x(n-m)\right\}$$

$$= -\sum_{m=1}^{N} a_m Z\{y(n-m)\} + \sum_{m=0}^{M} b_m Z\{x(n-m)\}$$

$\longrightarrow (1)$

let $y(n)=0$ for $n<0$, now if $Z\{y(n)\}=Y(z)$, then

$$Z\{y(n-m)\}=z^{-m}y(z)$$

using shifting property

let $x(n)=0$ for $n<0$, now if $Z\{x(n)\}=X(z)$, then

$$Z\{x(n-m)\}=z^{-m}x(z)$$

Using shifting property of Z-Transform, the eqn (1) is written as

$$Y(z)=-\sum_{m=1}^{N} a_m z^{-m} Y(z)+\sum_{m=0}^{M} b_m z^{-m} X(z).$$

$$Y(z)+\sum_{m=1}^{N} a_m z^{-m} Y(z)=\sum_{m=0}^{M} b_m z^{-m} X(z)$$

$$Y(z)\left[1+\sum_{m=1}^{N} a_m z^{-m}\right]=\sum_{m=0}^{M} b_m z^{-m} X(z).$$

$$\therefore \frac{Y(z)}{X(z)}=\frac{\sum_{m=0}^{M} b_m z^{-m}}{1+\sum_{m=1}^{N} a_m z^{-m}} \longrightarrow (2)$$

On expanding the summation on the above eqn (2) we get,

$$\frac{Y(z)}{X(z)}=\frac{b_0+b_1z^{-1}+b_2z^{-2}+\cdots\cdots+b_Mz^{-M}}{1+a_1z^{-1}+a_2z^{-2}+\cdots\cdots+a_Nz^{-N}} \longrightarrow (3)$$

The transfer function of a discrete time sys is defined as the ratio of Z-Transform of o/p and Z-Transform of input.

Hence the eqn (2) is the transfer function of an LTI discrete time sys.

The eqn (3) is a rational function of z^{-1} (i.e., ratio of two polynomials in z^{-1}). The numerator and denominator polynomials of eqn (3) are converted to positive power of z and then expressed in the factorized form as shown below.

$$\frac{Y(z)}{X(z)} = G \cdot \frac{(z-z_1)(z-z_2)(z-z_3)\ldots(z-z_N)}{(z-p_1)(z-p_2)(z-p_3)\ldots(z-p_N)}$$

Let, $M=N$ ⟶ (4)

where $z_1, z_2, z_3 \ldots z_N$ are roots of numerator polynomials (zeros of DT Sig)

$p_1, p_2, p_3 \ldots p_N$ are roots of denominator polynomial (poles of DT Sig).

Impulse response and Transfer function:-

Let, $x(n)$ = Input of an LTI Discrete time sys.

$y(n)$ = output (or) Response of the LTI discrete time sys for the i/p $x(n)$.

$h(n)$ = Impulse response (i.e. response for impulse i/p).

Now, the response $y(n)$ of the discrete time Slm is given by convolution of i/p and impulse response.

$$\therefore y(n) = x(n) * h(n) = \sum_{m=-\infty}^{\infty} x(m).h(n-m).$$

On talking Z-Transform of the above eqn, we get

$$Z\{y(n)\} = Z\{x(n) * h(n)\}$$

Using convolution property of Z-Transform, the above eqn can be written as.

$$Y(Z) = X(Z).H(Z)$$

$$\therefore H(Z) = \frac{Y(Z)}{X(Z)}.$$

$$\therefore H(Z) = \frac{Y(Z)}{X(Z)} = G.\frac{(Z-Z_1)(Z-Z_2)---(Z-Z_N)}{(Z-P_1)(Z-P_2)---(Z-P_N)}$$

from eqn ① we can conclude that ① is the transfer function of an LTI discrete time Slm is also given by Z-Transform of the impulse response.

Alternatively, the inverse Z-Transform of transfer function is the impulse response of the system.

$$\therefore \text{Impulse response, } h(n) = Z^{-1}\{H(Z)\} = Z^{-1}\left\{\frac{Y(Z)}{X(Z)}\right\}$$

Stability in Z-Domain:-

Location of poles for stability:-

Let, $h(n)$ be the impulse response of an LTI discrete time sim. Now, if $h(n)$ satisfies the condition,

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty \longrightarrow (1)$$

then the LTI discrete time sim is stable.

The stability condition of eqn (1) can be transformed as a condition on location of poles of transfer function of the LTI discrete time system in Z-plane.

Let $h(n) = a^n u(n)$

Now, $\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} |a^n u(n)| = \sum_{n=0}^{\infty} a^n$

If $|a|$ is $0 < |a| < 1$, then $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} = \text{constant}$, so the system is stable.

If $|a| > 1$, then $\sum_{n=0}^{\infty} a^n = \infty$ so sim is unstable.

Now, $H(z) = Z\{h(n)\} = Z\{a^n u(n)\} = \frac{z}{z-a}$.

Here $H(z)$ has pole at $z = a$.

If $|a| < 1$, then pole will lie inside the unit circle

$(z-1/4)(z-1/2)$

and if $|a|>1$, then the pole will be lie outside the unit circle.

∴ For a stable discrete time system the poles should be lie inside the unit circle.

* Determine the pole-zero plot and pole-zero stability for the described by difference equation $y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n) - x(n-1)$.

Sol:- Given difference equation

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n) - x(n-1)$$

Taking Z-Transform on both-sides, we obtain

$$Y(z) - \frac{3}{4}z^{-1}Y(z) + \frac{1}{8}z^{-2}Y(z) = X(z) - z^{-1}X(z)$$

$$Y(z)\left[1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right] = X(z)\left[1 - z^{-1}\right]$$

The system function

$$\frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

$$= \frac{1 - z^{-1}}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}$$

$$= \frac{z(z-1)}{(z - \frac{1}{4})(z - \frac{1}{2})} \qquad \text{ROC: } |z| > \frac{1}{2}$$

Zeros are at $z=0,1$; Poles are at $z=1/4, 1/2$.

The pole-zero plot is

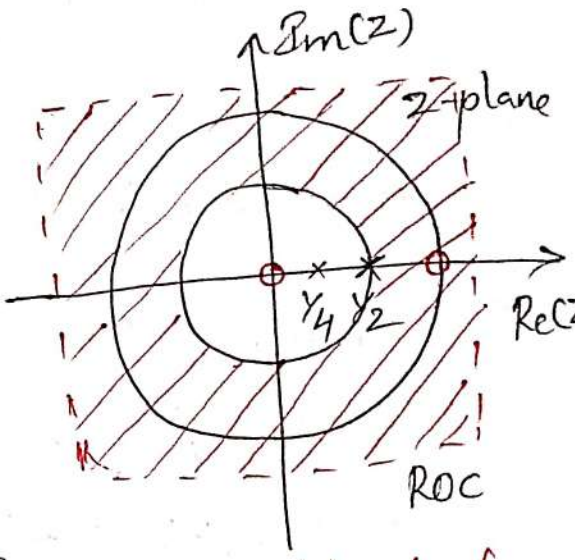


fig: Pole-zero plot of ~~fig~~.

From the figure we can observe the following

1) ROC of the Sln function includes unit circle.
2) ROC of the Sln function cannot contain any poles.

∴ The system ~~is not~~ a stable.

(2) Find the pole-zero plot for the system described by diffence equation. & check the pole-zero stability

$$y(n) = 5/6\, y(n-1) - 1/6\, y(n-2) + x(n) - x(n-1)$$

Sol:- Given $y(n) = 5/6\, y(n-1) - 1/6\, y(n-2) + x(n) - x(n-1)$

Taking Z-Transform on both sides

$$Y(z) = 5/6\, z^{-1} Y(z) - 1/6\, z^{-2} Y(z) + X(z) - z^{-1} X(z)$$

$$Y(z) - 5/6\, z^{-1} Y(z) + 1/6\, z^{-2} Y(z) = X(z) - z^{-1} X(z)$$

$$Y(z)\left[1 - 5/6\, z^{-1} + 1/6\, z^{-2}\right] = X(z)\left[1 - z^{-1}\right]$$

$$\frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 - 5/6\, z^{-1} + 1/6\, z^{-2}}$$

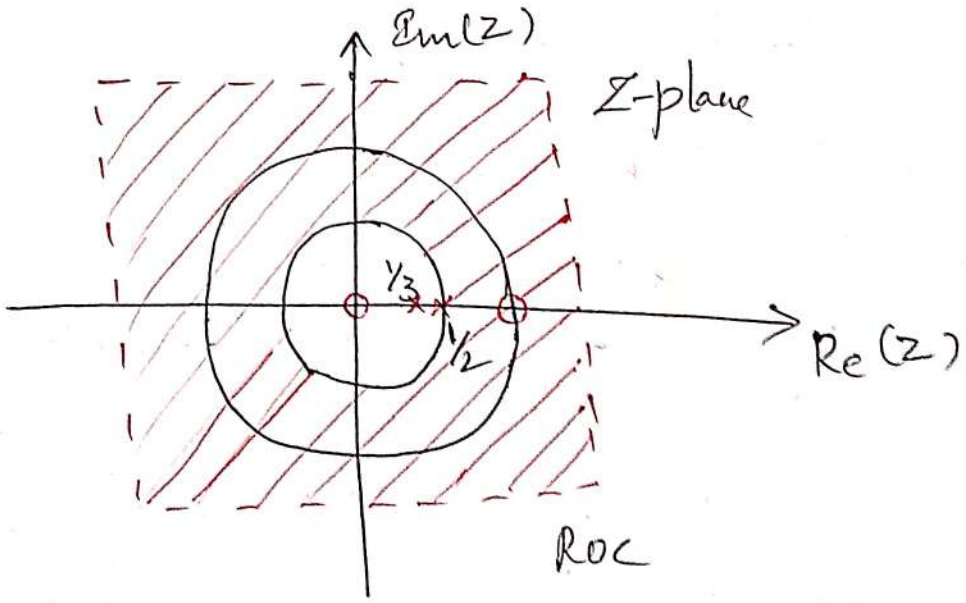
The system function

$$\frac{Y(z)}{X(z)} = \frac{1-z^{-1}}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}$$

$$= \frac{1-z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}$$

$$= \frac{1-z^{-1}}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$$

Zeros are at $z=0, 1$; poles are at $z=\frac{1}{2}, \frac{1}{3}$.



1. The ROC of the system function includes unit circle.
2. The ROC of Slm function cannot contain any poles.

∴ The Slm is ~~not~~ stable.

Solution to difference equation with initial conditions:-

In a linear constant coefficient difference equation, which involved finding the particular and homogenous solution. Instead we use the time shifting property of Z-Transform to the difference equation. So we can solve the transform of the o/p. Then applying Inverse transform we can find the output sequence.

(1) Find the impulse response of the system described by difference equation
$y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1)$ using Z-Transform.

(A) Given $y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1)$

Applying Z-Transform on both sides.

$$Y(z) - 3z^{-1}Y(z) - 4z^{-2}Y(z) = X(z) + 2z^{-1}X(z).$$

$$Y(z)[1 - 3z^{-1} - 4z^{-2}] = X(z)[1 + 2z^{-1}]$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 2z^{-1}}{1 - 3z^{-1} - 4z^{-2}}$$

$$H(z) = \frac{z(z+2)}{z^2-3z-4}$$

$$\frac{H(z)}{z} = \frac{z+2}{z^2-3z-4}$$

$$= \frac{z+2}{(z-4)(z+1)}$$

$$= \frac{A}{(z-4)} + \frac{B}{z+1}$$

$$A = (z-4)\cdot \frac{z+2}{(z-4)(z+1)}\Big|_{z=4}$$

$$A = \frac{4+2}{4+1} = 6/5$$

$$B = (z+1)\cdot \frac{z+2}{(z-4)(z+1)}\Big|_{z=-1}$$

$$B = \frac{-1+2}{-1-5} = -1/5$$

$$\therefore H(z) = (6/5)\cdot\frac{z}{z-4} - (1/5)\frac{z}{z+1}$$

$$\therefore h(n) = 6/5\,(4)^n u(n) - 1/5\,(-1)^n u(n)$$

(2) A causal LTI system is described by the difference equation $y(n) = y(n-1) + y(n-2) + x(n-1)$ where $x(n)$ is the i/p & $y(n)$ is the o/p. Find

(a) The system function $H(z) = \frac{Y(z)}{X(z)}$ for the

System. Plot the poles and zeros of H(z) & indicate the ROC.

(b) Find the unit sample response of the sim.

(c) Is the system is stable (or) Not?

(A) Given $y(n) = y(n-1) + y(n-2) + x(n-1)$.

Apply Z-Transform on both sides.

$$Y(z) = z^{-1}Y(z) + z^{-2}Y(z) + z^{-1}X(z)$$

$$Y(z) - z^{-1}Y(z) + z^{-2}Y(z) = z^{-1}X(z)$$

$$Y(z)\left[1 - z^{-1} - z^{-2}\right] = z^{-1}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - z^{-1} - z^{-2}}$$

$$H(z) = \frac{z}{z^2 - z - 1}$$

$x = \frac{-b \pm \sqrt{b^2 - 4}}{2a}$

(B) The poles are at $\frac{1 \pm \sqrt{5}}{2}$ and zeros at $z = 0$. The pole-zero diagram as shown in below.

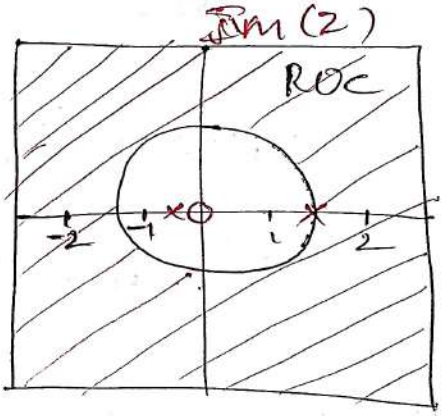


fig: Pole zero pattern and ROC. of

$$\frac{H(z)}{z} = \frac{1}{z^2 - z - 1}$$

(A)

$$\frac{H(z)}{z} = \frac{1}{(z + 0.618)(z - 1.618)}$$

$$= \frac{A}{z - 1.618} + \frac{B}{z + 0.618}$$

$$A = (z-0.618)\left.\frac{1}{(z+0.618)(z-0.618)}\right|_{z=1.618}$$

$$= \frac{1}{1.618+0.618} = \frac{1}{2.236} = 0.447$$

$$B = (z+0.618)\left.\frac{1}{(z-1.618)(z+0.618)}\right|_{z=-0.618}$$

$$= \frac{1}{-0.618-1.618} = \frac{-1}{2.236} = -0.447$$

$$\frac{H(z)}{z} = \frac{0.447}{z-0.618} - \frac{0.447}{z+0.618}$$

$$\therefore H(z) = \frac{z(0.447)}{z-1.618} - \frac{z(0.447)}{z+0.618}$$

$$h(n) = 0.447(1.618)^n u(n) - 0.447(-0.618)^n u(n)$$

∴ The pole $z = 1.618$ lies outside the unit circle hence the system is unstable.

(3) Determine the unit step response of the system whose difference equation is $y(n) - 0.7y(n-1) + 0.12y(n-2) = x(n-1) + x(n-2)$ if $y(-1) = y(-2) = 1$.

(A) Given $y(n) - 0.7y(n-1) + 0.12y(n-2) = x(n-1) + x(n-2)$.

Taking Z-Transform on both sides we have.

$$Y(z) - 0.7z^{-1}Y(z) + 0.12z^{-2}Y(z) = z^{-1}X(z) + z^{-2}X(z)$$

$$Y(z) - 0.7[z^{-1}Y(z) + y(-1)] + 0.12[z^{-2}Y(z) + z^{-1}y(-1) + y(-2)].$$

$$= z^{-1}X(z) + x(-1) + z^{-2}X(z) + z^{-1}x(-1) + x(-2)$$

Substituting initial conditions $y(-1) = y(-2) = 1$ and $x(-1) = x(-2) = 0$

∴ for a step input $x(-1) = x(-2) = 0$

$$Y(z) - 0.7[z^{-1}Y(z) + 1] + 0.12[z^{-2}Y(z) + z^{-1} + 1] =$$

$$z^{-1}X(z) + z^{-2}X(z)$$

for unit step input $X(z) = \frac{1}{1-z^{-1}}$

$$Y(z)[1 - 0.7z^{-1} + 0.12z^{-2}] - 0.58 + 0.12z^{-1} = \frac{z^{-1}}{1-z^{-1}} + \frac{z^{-2}}{1-z^{-1}}$$

$$Y(z) = \frac{z^{-1} + z^{-2}}{(1-z^{-1})(1 - 0.7z^{-1} + 0.12z^{-2})} + \frac{0.58 - 0.12z^{-1}}{1 - 0.7z^{-1} + 0.12z^{-2}}$$

$$= \frac{z(z+1)}{(z-1)(z-0.4)(z-0.3)} + \frac{0.58z^2 - 0.12z}{(z-0.4)(z-0.3)}$$

$$\frac{Y(z)}{z} = \frac{(z+1)}{(z-1)(z-0.4)(z-0.3)} + \frac{0.58z - 0.12}{(z-0.4)(z-0.3)}$$

$$= \frac{11.762}{z-1} - \frac{23.33}{z-0.4} + \frac{18.57}{z-0.3} + \frac{1.12}{z-0.4} - \frac{0.54}{z-0.3}$$

Multiplying z on both sides & taking inverse-Z-Transform we have $y(n) = 11.76\,u(n) - 22.21(0.4)^n u(n) + 18.03(0.3)^n u(n)$

(11) Determine the impulse response h(n) for the LTI system described by the second-order difference equation. $y(n)+4y(n-1)+3y(n-2)=x(n-1)$.

(A) The difference equation of the LTI system is

$$y(n)+4y(n-1)+3y(n-2)=x(n-1).$$

Let us take Z-Transform of the difference equation governing the LTI system with zero-initial-conditions.

$$Y(z)+4z^{-1}Y(z)+3z^{-2}Y(z)=z^{-1}X(z)$$

$$Y(z)\left[1+4z^{-1}+3z^{-2}\right]=z^{-1}X(z)$$

$$\therefore \frac{Y(z)}{X(z)}=\frac{z^{-1}}{1+4z^{-1}+3z^{-2}}$$

W.K.T. $H(z)=\frac{Y(z)}{X(z)}$

$$\therefore H(z)=\frac{Y(z)}{X(z)}=\frac{z^{-1}}{1+4z^{-1}+3z^{-2}}$$

$$=\frac{z^{-1}}{z^{-2}\left[z^2+4z+3\right]}=\frac{z}{(z+1)(z+3)}.$$

Using partial fraction expansion technique

$$\frac{H(z)}{z}=\frac{1}{(z+1)(z+3)}=\frac{A}{z+1}+\frac{B}{z+3}$$

$$A = (z+1)\frac{1}{(z+1)(z+3)}\Big|_{z=-1} = \frac{1}{-1+3} = \frac{1}{2} = 0.5$$

$$B = (z+3)\frac{1}{(z+1)(z+3)}\Big|_{z=-3} = \frac{1}{-3+1} = -\frac{1}{2} = -0.5$$

$$\therefore \frac{H(z)}{z} = \frac{0.5}{z+1} - \frac{0.5}{z+3}$$

$$H(z) = \frac{0.5z}{z+1} - \frac{0.5z}{z+3}$$

The impulse response $h(n)$ is given by inverse Z-transform of $H(z)$.

Impulse response $h(n)$ is

$$h(n) = z^{-1}\{H(z)\}$$

$$= z^{-1}\left\{\frac{0.5z}{z+1} - \frac{0.5z}{z+3}\right\}$$

$$= 0.5z^{-1}\left\{\frac{z}{z-(-1)}\right\} - 0.5z^{-1}\left\{\frac{z}{z-(-3)}\right\}$$

$$= 0.5(-1)^n u(n) - 0.5(-3)^n u(n)$$

$$h(n) = 0.5[(-1)^n - (-3)^n]u(n)$$

(5) Find the impulse response of the system described by difference equation.

$$y(n) - 3y(n-1) + 4y(n+2) = x(n) - 5x(n-3)$$

(A) Given LTI sim is

$$y(n) - 4y(n-1) + 4y(n-2) = x(n) - 5x(n-3).$$

Taking Z-Transform on both sides

$$Y(z) - 4z^{-1}Y(z) + 4z^{-2}Y(z) = X(z) - 5z^{-3}X(z)$$

$$Y(z)\left[1 - 4z^{-1} + 4z^{-2}\right] = X(z)\left[1 - 5z^{-3}\right]$$

$$\therefore \frac{Y(z)}{X(z)} = \frac{1 - 5z^{-3}}{1 - 4z^{-1} + 4z^{-2}}$$

$$\therefore H(z) = \frac{z^2\left[1 - 5z^{-3}\right]}{z^{-2}\left[1 - 4z^{-1} + 4z^{-2}\right]} = \frac{z^2 - 5z^{-1}}{z^2 - 4z + 4}.$$

$$H(z) = \frac{z^2 - 5z^{-1}}{(z-2)^2}$$

$$\Rightarrow \frac{z^2}{(z-2)^2} - \frac{5z^{-1}}{(z-2)^2}$$

$$\Rightarrow \frac{1}{2} z \cdot \frac{2z}{(z-2)^2} - \frac{5}{2} z^{-2} \cdot \frac{2z}{(z-2)^2}$$

The impulse response is obtained by taking inverse Z-Transform of $H(z)$.

$\therefore$ Impulse response, $h(n) = Z^{-1}\{H(z)\}$

$$\Rightarrow Z^{-1}\left\{\frac{1}{2} z \cdot \frac{2z}{(z-2)^2} - \frac{5}{2} z^{-2} \cdot \frac{2z}{(z-2)^2}\right\}.$$

$$\therefore h(n) \Rightarrow \frac{1}{2}(n+1)^{n+1} u(n+1) + \frac{5}{2}(n-2)(2)^{n-2} u(n-2).$$

(5) Obtain and sketch the impulse response of Shift invariant Sm described by

$$y(n) = 0.4x(n) + x(n-1) + 0.2x(n-2) + x(n-3) + 0.6x(n-4)$$

(A) Given $y(n) = 0.4x(n) + x(n-1) + 0.2x(n-2) + x(n-3) + 0.6x(n-4)$.

On taking Z-Transform on both sides.

$$Y(z) = 0.4X(z) + z^{-1}X(z) + 0.2z^{-2}X(z) + z^{-3}X(z) + 0.6z^{-4}X(z)$$

$$Y(z) = [0.4 + z^{-1} + 0.2z^{-2} + z^{-3} + 0.6z^{-4}]X(z)$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = 0.4 + 1.2^{-1} + 0.2z^{-2} + z^{-3} + 0.6z^{-4} \longrightarrow (1)$$

By the definatin of on sided Z-Transform. we get

$$\therefore H(z) = \sum_{n=0}^{\infty} h(n) \cdot z^{-n}$$

$$= h(0)z^0 + h(1)z^1 + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4}. \longrightarrow (2)$$

On comparing eqn (1) & (2) we get

$h(0) = 0.4$, $h(1) = 1$, $h(2) = 0.2$, $h(3) = 1$, $h(4) = -0.6z^{-4}$

$\therefore$ Impulse response, $h(n) = \{0.4, 1, 0, 0.2, 1.0, 0.6\}$

— RND —

UNIT- II

DISCRETE FOURIER TRANSFORM.

* Discrete Fourier Series and their properties.
* Discrete Fourier Transform and properties of DFT.
* Problems on DFT.
* Convolution using DFT and related problems.

Fast Fourier Transform :-

* Introduction to FFT.
* Radix - 2 Decimation in Time - (DIT-FFT) Algorithm.
  (a) Butterfly diagram.
* Radix - 2 Decimation in frequency - (DIF - FFT) Algorithm.
* Problems related to DIT & DIF - FFT.
* Inverse FFT using radix - 2 Algorithm and related problems.

DISCRETE FOURIER TRANSFORM

Discrete Fourier Series:-

* Fourier series is applicable only for periodic signal.
* The discrete fourier series is an alternative representation of a periodic sequence $x(n)$ with period N. The periodic sequence $x(n)$ can be represented as a sum of N complex exponentials with frequency, $\frac{2\pi k}{N}$.

Where,

$k = 0$ to $(N-1)$.

* The discrete fourier series of $x(n)$ can be represented as,

$$x(n) = \sum_{k=0}^{N-1} C_k \cdot e^{\frac{jk2\pi n}{N}}$$

Where, C_k is a "discrete fourier series" co-efficient and is given by,

$$C_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) \cdot e^{\frac{-jk2\pi n}{N}}$$

Properties of Discrete Fourier Series:-

1. Linear property:-

If $x_1(n) \xleftrightarrow{DFS} C_k$ & $x_2(n) \xleftrightarrow{DFS} D_k$. Then,

$$DFS[a x_1(n) + b x_2(n)] = a C_k + b D_k$$

Proof:-

By the defination of DFS we have,

$$C_k = DFS[x(n)] = \frac{1}{N} \sum_{n=0}^{N-1} x(n) \cdot e^{\frac{-jk2\pi n}{N}} \longrightarrow (1)$$

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$$DFS[ax_1(n)+bx_2(n)] = \frac{1}{N}\sum_{n=0}^{N-1}[ax_1(n)+bx_2(n)].e^{-\frac{jk2\pi n}{N}}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} ax_1(n).e^{-\frac{jk2\pi n}{N}} + \frac{1}{N}\sum_{n=0}^{N-1} bx_2(n).e^{-\frac{jk2\pi n}{N}}$$

$$= a\left[\frac{1}{N}\sum_{n=0}^{N-1} x_1(n).e^{-\frac{jk2\pi n}{N}}\right] + b\left[\frac{1}{N}\sum_{n=0}^{N-1} x_2(n)e^{-\frac{jk2\pi n}{N}}\right]$$

$$= aC_k + bD_k = R.H.S$$

$\therefore$ L.H.S = R.H.S.

$$DFS[ax_1(n)+bx_2(n)] = aC_k + bD_k$$

2. Time Shifting property:-

If $x(n) \xleftrightarrow{DFS} C_k$, then $x(n-m) \xleftrightarrow{DFS} C_k.e^{-\frac{jk2\pi}{N}m}$

Proof:- By using defination of DFS we have,

$$C_k = DFS[x(n)] = \frac{1}{N}\sum_{n=0}^{N-1} x(n).e^{-\frac{jk2\pi n}{N}} \longrightarrow (1)$$

$$DFS[x(n-m)] = \frac{1}{N}\sum_{n=0}^{N-1} x(n-m).e^{-\frac{jk2\pi n}{N}}$$

Let $n-m=p$, if $n=0 \Rightarrow p=-m$

$n=p+m$ $\qquad$ $n=N-1 \Rightarrow p=N-1-m$

$$= \frac{1}{N}\sum_{p=-m}^{N-1-m} x(p).e^{-\frac{jk2\pi}{N}(p+m)}$$

$$= \frac{1}{N}\sum_{p=-m}^{N-1-m} x(p).e^{-\frac{jk2\pi}{N}p}.e^{-\frac{jk2\pi}{N}.m}$$

$$= C_k.e^{-\frac{jk2\pi}{N}.m}$$

$\therefore$ L.H.S = R.H.S

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3. Frequency shifting property:-

If $x(n) \xleftrightarrow{DFS} C_k$, then $x(n)e^{\frac{jl2\pi}{N}.n} \xleftrightarrow{DFS} C_{k-l}$.

Proof:- By the defination of DFS we have,

$$C_k = DFS[x(n)] = \frac{1}{N}\sum_{n=0}^{N-1} x(n).\ e^{\frac{-jk2\pi n}{N}} \longrightarrow (1)$$

$$DFS\left[x(n)\,e^{\frac{-jl2\pi}{N}n}\right] = \frac{1}{N}\sum_{n=0}^{N-1} x(n).\ e^{\frac{jl2\pi}{N}n}.e^{\frac{-jk2\pi}{N}n}.$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} e^{-j(k-l)\frac{2\pi}{N}n}.\ x(n)$$

$$= C_{k-l}.$$

$\therefore$ LHS = RHS

4. Convolution property:-

If $x_1(n) \xleftrightarrow{DFS} C_k$ & $x_2(n) \xleftrightarrow{DFS} D_k$, then

$$DFS[x_1(n) * x_2(n)] = N.C_k.D_k.$$

Proof:- By the defination of DFS we have,

$$C_k = DFS[x(n)] = \frac{1}{N}\sum_{n=0}^{N-1} x(n).\ e^{\frac{-jk2\pi n}{N}} \longrightarrow (1)$$

$$x_1(n) * x_2(n) = \sum_{m=0}^{N-1} x(m).\ x(n-m)$$

$$DFS[x_1(n) * x_2(n)] = \frac{1}{N}\sum_{n=0}^{N-1}\left[\sum_{m=0}^{N-1} x(m).x(n-m)\right] e^{\frac{-jk2\pi n}{N}}$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} x(m).\left[\sum_{n=0}^{N-1} x(n-m).\ e^{\frac{-jk2\pi}{N}.n}\right]$$

Let $n-m=p$, if $n=0 \Rightarrow p=-m$

$n=p+m$, $n=N-1 \Rightarrow p=N-1-m$.

$$= \frac{1}{N}\sum_{m=0}^{N-1} x(m)\left[\sum_{p=-m}^{N-1-m} x(p).e^{\frac{-jk2\pi}{N}(p+m)}\right]$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} x(m)\left[\sum_{p=-m}^{N-1-m} x(p).\ e^{\frac{-jk2\pi}{N}.p}.e^{\frac{-jk2\pi}{N}m}\right]$$

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$$= \frac{N}{N} \cdot \frac{1}{N} \sum_{m=0}^{N-1} x(m) \cdot e^{-\frac{jk2\pi}{N} \cdot m} \sum_{p=-m}^{N-1-m} x(p) \cdot e^{-\frac{jk2\pi m}{N} \cdot p}$$

$$= N \cdot \frac{1}{N} \sum_{m=0}^{N-1} x(m) \cdot e^{-\frac{jk2\pi}{N} m} \frac{1}{N} \sum_{p=-m}^{N-1-m} x(p) \cdot e^{-\frac{jk2\pi}{N} \cdot p}$$

$$= N \cdot C_k \cdot D_k$$

$\therefore$ LHS = RHS

5. Multiplication property:-

If $x_1(n) \xleftrightarrow{DFS} C_k$ & $x_2(n) \xleftrightarrow{DFS} D_k$ then

$$DFS\left[x_1(n) \cdot x_2(n)\right] = \sum_{l=0}^{N-1} C_l D_{k-l}$$

Proof:- By the defination of DFS we have,

$$C_k = DFS[x(n)] = \frac{1}{N} \sum_{n=0}^{N-1} x(n) \cdot e^{-\frac{jk2\pi n}{N}} \rightarrow (1)$$

$$DFS\left[x_1(n) \cdot x_2(n)\right] = \frac{1}{N} \sum_{n=0}^{N-1} \left[x_1(n) \cdot x_2(n)\right] \cdot e^{-\frac{jk2\pi}{N} \cdot n}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \left[\sum_{l=0}^{N-1} C_l \cdot e^{-\frac{jl2\pi n}{N}} \cdot x_2(n) \cdot e^{-\frac{jk2\pi}{N} \cdot n}\right]$$

$$\left(\because x(n) = \sum_{K=0}^{N-1} C_k \cdot e^{\frac{jk2\pi n}{N}}\right)$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} x_2(n) \cdot e^{-\frac{jk2\pi n}{N}} \cdot \sum_{l=0}^{N-1} C_l \cdot e^{-\frac{jl2\pi}{N} \cdot n}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} x_2(n) \cdot e^{-j(k-l)\frac{2\pi}{N} \cdot n} \cdot \sum_{l=0}^{N-1} C_l$$

$$= \sum_{l=0}^{N-1} C_l \cdot D_{k-l}$$

$\therefore$ LHS = RHS.

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6. Conjugate property:-

If $x(n) \xleftrightarrow{DFS} X(e^{jw})$ then

$$F[x(n)] = X^*(e^{-jw})$$

Proof:- By the defination of DFS we have,

$$X(e^{jw}) = F[x(n)] = \sum_{n=-\infty}^{\infty} x(n).e^{-jw}$$

$$F[x(n)] = \sum_{n=-\infty}^{\infty} x^*(n).e^{-jwn}$$

$$= \left[\sum_{n=-\infty}^{\infty} x(n)\,(e^{-jw})^{-n}\right]^*$$

$$= [x(e^{-jw})]^*$$

$$= x^*(e^{-jw})$$

$\therefore$ LHS = RHS.

Discrete Fourier Transforms:-

* The DFT is a powerful computational tool which allows us to evaluate the fourier transform $X(e^{jw})$ on a digital computer (or) specially designed hardware.

* Unlike DTFT which is designed for finite (or) the infinite sequences. DFT is designed only for the sequences of finite length. Since $X(e^{jw})$ is the continuous and periodic, DFT is obtained by sampling one period of fourier transform at a finite number of frequency points.

* DFT plays an important role in the implementation of many signal processing algorithms. Apart from determining the frequency content of a signal, DFT

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is used to perform linear filtering operations in the frequency domain.

* The DFT of a sequence $x(n)$ is define as X_k.

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{\frac{-j2\pi nk}{N}} \longrightarrow (1)$$

Where,

$k = 0, 1, 2, 3, \ldots\ldots N-1$.

* The inverse DFT of a $X(k)$ is defined as:

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \, e^{\frac{j2\pi nk}{N}} \; ; n = 0, 1, 2, \ldots\ldots N-1$$

$\longrightarrow (2)$

- Twiddle Factor (W_N):-

Twiddle factors are set of values that is used to speed up DFT & IDFT calculations.

$$\boxed{e^{\frac{-j2\pi}{N}} = W_N}$$

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{nk} \longrightarrow (1)$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot W_n^{-nk} \longrightarrow (2)$$

$nk = r$	$W_8^r = e^{\frac{-j2\pi}{N}}$
0	$W_8^0 = 1$
1	$W_8^1 = e^{\frac{-j2\pi}{8} \cdot 1} = e^{\frac{-j\pi}{4}} = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$ $= 0.707 - j0.707$
2	$W_8^2 = e^{\frac{-j\pi}{4} \cdot 2} = e^{-j\pi/2} = \cos\pi/2 - j\sin\pi/2$ $= 0 - j1$ $= -j$

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3 $W_8^3 = e^{\left(\frac{-j\pi}{4}\right).3} = e^{-j3\pi/4} = \cos 3\pi/4 - j\sin 3\pi/4$
$= -0.707 - j0.707$

4 $W_8^4 = e^{\left(\frac{-j\pi}{4}\right).4} = e^{-j\pi} = \cos\pi - j\sin\pi = -1 - j0 = -1.$

5 $W_8^5 = e^{\left(\frac{-j\pi}{4}\right).5} = e^{\frac{-j5\pi}{4}} = \cos 5\pi/4 - j\sin 5\pi/4$
$= -0.707 + j0.707.$

6. $W_8^6 = e^{\left(\frac{-j\pi}{4}\right).6^3} = e^{\frac{-j3\pi}{2}} = \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}$
$= 0 - j(-1)$
$= +j$

7 $W_8^7 = e^{\left(\frac{-j\pi}{4}\right).7} = e^{\frac{-j7\pi}{4}} = \cos 7\pi/4 - j\sin 7\pi/4$
$= 0.707 + j0.707$

$W_8^0 = 1$

$W_8^1 = 0.707 - j0.707$

$W_8^2 = -j$

$W_8^3 = -0.707 - j0.707$

$W_8^4 = -1$

$W_8^5 = -0.707 + j0.707$

$W_8^6 = +j$

$W_8^7 = 0.707 + j0.707$

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Zero Padding:-

Let the sequence $x(n)$ as a length L. If we want to find N-point DFT of a sequence $x(n)$ where $N>L$. We have to add $(N-L)$ zeroes to the sequence $x(n)$. This is known as zero padding.

Uses of Zero padding:-

1. We can get better display of the frequency spectrum.
2. The DFT with zero padding can be used in Linear filtering.

Properties of DFT:-

1. Linearity property:-

If $x_1(n) \xleftrightarrow{DFT} X_1(k)$ & $x_2(n) \xleftrightarrow{DFT} X_2(k)$ then,

$$[ax_1(n)+bx_2(n)] \xleftrightarrow{DFT} aX_1(k)+bX_2(k)$$

Proof:- By using the defination of DFT we have, $DFT[x(n)]=X(k)=\sum_{n=0}^{N-1} x(n).e^{\frac{-j2\pi nk}{N}}$

$$DFT[ax_1(n)+bx_2(n)]=\sum_{n=0}^{N-1}[ax_1(n)+bx_2(n)].e^{\frac{-j2\pi nk}{N}}$$

$$=\sum_{n=0}^{N-1}(ax_1(n).e^{\frac{-j2\pi nk}{N}}+\sum_{n=0}^{N-1}bx_2(n).e^{\frac{-j2\pi nk}{N}}$$

$$=a\sum_{n=0}^{N-1}x_1(n).e^{\frac{-j2\pi nk}{N}}+b\sum_{n=0}^{N-1}x_2(n).e^{\frac{-j2\pi nk}{N}}$$

$$=aX_1(k)+bX_2(k)$$

$\therefore$ L.H.S = R.H.S

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2. Circular Time shifting property:-

If $x(n) \xleftrightarrow{DFT} X(k)$ then,

$x((n-m)_N) \xleftrightarrow{DFT} e^{-j2\pi km/N} X(k)$.

Proof:- By defination $DFT[x(n)] = X(k) = \sum_{n=0}^{N-1} x(n).e^{-j2\pi nk/N}$

$DFT[x(n-m)_N] = \sum_{n=0}^{N-1} x((n-m)_N).e^{-j2\pi nk/N}$

As we know $x[(n-m)_N] = x[N+n-m]$

$= \sum_{n=0}^{N-1} x(N+n-m).e^{-j2\pi nk/N}$

Let,

$N+n-m=p$

$n = p+m-N$

Limit

At $n=0$; $p = N-m$

At $n = N-1$; $p = N+N-1-m$

$p = 2N-1-m$.

$= \sum_{p=N-m}^{2N-1-m} x(p).e^{-j2\pi k/N}(p+m-N)$.

$= \sum_{p=N-m}^{2N-1-m} x(p).e^{-j2\pi kp/N}.e^{-j2\pi km/N}.e^{j2\pi k(1-N)/N}$

$= \sum_{p=N-m}^{2N-1-m} x(p).e^{-j2\pi kp/N}.e^{-j2\pi km/N}.e^{-j2\pi k}$

$\because [e^{j2\pi k} = 1]$

$= e^{-j2\pi km/N}.X(k)$

$\therefore$ L.H.S = R.H.S

3. Circular Frequency shifting property:-

If $x(n) \xleftrightarrow{DFT} X(k)$ then,

$e^{+j2\pi kl/N}.x(n) \xleftrightarrow{DFT} X((k-l)_N)$

Proof:- By the defination of DFT,

$DFT[x(n)] = X(k) = \sum_{n=0}^{N-1} x(n).e^{-j2\pi nk/N}$

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$$DFT\left[e^{+\frac{j2\pi lk}{N}} \cdot x(n)\right] = \sum_{n=0}^{N-1} \left(e^{+\frac{j2\pi ln}{N}} \cdot x(n)\right) \cdot e^{-\frac{j2\pi nk}{N}}$$

$$= \sum_{n=0}^{N-1} x(n) \cdot e^{-\frac{j2\pi nk}{N}} \cdot e^{+\frac{j2\pi nl}{N}} \cdot e^{-j\frac{N2\pi n}{N}}$$

$$= \sum_{n=0}^{N-1} x(n) \cdot e^{-j(N+k-l)\frac{2\pi n}{N}}$$

As we know $x((n-m)_N) = x(N+n-m)$

$$= x((k-l)_N)$$

$\therefore$ L.H.S = R.H.S

4. Circular Convolution property:-

If $x_1(n) \xleftrightarrow{DFT} X_1(k)$ & $x_2(n) \xleftrightarrow{DFT} X_2(k)$, then

$x_1(n) \circledast x_2(n) \xleftrightarrow{DFT} X_1(k) \cdot X_2(k)$.

Proof:-

By defination, $X(k) = DFT[x(n)] = \sum_{n=0}^{N-1} x(n) \cdot e^{-\frac{j2\pi nk}{N}}$

$$x_1(n) \circledast x_2(n) = \sum_{m=0}^{N-1} x(m) \cdot x_2((n-m)_N)$$

$$DFT[x_1(n) \circledast x_2(n)] = \sum_{n=0}^{N-1} (x_1(n) \circledast x_2(n)) e^{-\frac{j2\pi nk}{N}}$$

$$= \sum_{n=0}^{N-1} \left(\sum_{m=0}^{N-1} x_1(m) \cdot x_2((n-m)_N) \cdot e^{-\frac{j2\pi nk}{N}} \right.$$

$$= \sum_{n=0}^{N-1} \left(\sum_{m=0}^{N-1} x_1(m) \cdot x_2(N+n-m) \right) \cdot e^{-\frac{j2\pi nk}{N}}$$

Interchange the summation,

$$\sum_{m=0}^{N-1} x_1(m) \left[\sum_{n=0}^{N-1} x_2(N+n-m)\, e^{-\frac{j2\pi nk}{N}} \right]$$

Let $N+n-m=p$ if $n=0$; $p = N-m$

$n = p+m-N$ if $n=N-1$; $p = 2N-1-m$.

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$$= \sum_{m=0}^{N-1} x_1(m) \left[\sum_{p=N-m}^{2N-1-m} x_2(p) \cdot e^{-\frac{j2\pi k}{N}(p+m-N)} \right]$$

$$= \sum_{m=0}^{N-1} x_1(m) \left[\sum_{p=N-m}^{2N-1-m} x_2(p) \cdot e^{-\frac{j2\pi pk}{N}} \cdot e^{-\frac{j2\pi km}{N}} \cdot e^{-\frac{j2\pi k(-N)}{N}} \right]$$

$$= \sum_{m=0}^{N-1} x_1(m) \cdot e^{-\frac{j2\pi mk}{N}} \cdot \sum_{p=N-m}^{2N-1-m} x_2(p) \cdot e^{-\frac{j2\pi pk}{N}} \qquad (\because e^{-j2\pi k} = 1)$$

$$= X_1(k) \cdot X_2(k)$$

$\therefore$ L.H.S = R.H.S

5. Multiplication property:-

If $x_1(n) \xleftrightarrow{DFT} X_1(k)$ & $x_2(n) \xleftrightarrow{DFT} X_2(k)$ then

$x_1(n) \cdot x_2(n) \xleftrightarrow{DFT} \frac{1}{N} [X_1(k) \circledast X_2(k)]$.

Proof:-

By defination, $X(k) = DFT[x(n)] = \sum_{n=0}^{N-1} x(n) \cdot e^{-\frac{j2\pi nk}{N}}$

$DFT[x_1(n) \cdot x_2(n)] = \sum_{n=0}^{N-1} (x_1(n) \cdot x_2(n)) e^{-\frac{j2\pi nk}{N}}$

As per IDFT $\left(x_1(n) = \frac{1}{N} \sum_{m=0}^{N-1} X_1(m) \cdot e^{-\frac{j2\pi nm}{n}} \right)$

$$= \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{m=0}^{N-1} x_1(m) \cdot e^{\frac{j2\pi mn}{N}} \right) \cdot x_2(n) \cdot e^{-\frac{j2\pi nk}{N}}$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} x_1(m) \left[\sum_{n=0}^{N-1} x_2(n) \cdot e^{-\frac{j2\pi nk}{N}} \cdot e^{\frac{j2\pi nm}{N}} \cdot e^{-\frac{j2\pi nN}{N}} \right]$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} X_1(m) \left[\sum_{n=0}^{N-1} x_2(n) e^{-\frac{j2\pi n}{N}(N+k-m)} \right]$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} X_1(m) \cdot X_2(N+k-m)$$

$$= \frac{1}{N} \left[\sum_{m=0}^{N-1} X_1(m) \cdot X_2((k-m)_N) \right]$$

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$$= \frac{1}{N}\left[X_1(m) \circledast X_2(k)\right]$$

$$\because \sum_{m=0}^{N-1} X_1(m).\, X_2\left((k-m)_N\right) = X_1(m) \circledast X_2(k)$$

6. Conjugate property:-

If $x(n) \xleftrightarrow{DFT} X(k)$ then

$$x^*(n) \xleftrightarrow{DFT} X^*\left((-k)_N\right).$$

Proof:-

By definition, $X(k) = DFT\left[x(n)\right] = \sum_{n=0}^{N-1} x(n).\, e^{\frac{-j2\pi nk}{N}}$

$$DFT\left[x^*(n)\right] = \sum_{n=0}^{N-1} x^*(n).\, e^{\frac{-j2\pi nk}{N}}$$

$$= \left[\sum_{n=0}^{N-1} x(n).\, e^{\frac{-j2\pi nk}{N}}\right]^*$$

$$= \sum_{n=0}^{N-1} x(n).\, e^{\frac{j2\pi nk}{N}}.\, e^{\frac{-j2\pi N.n}{N}}$$

$$= \sum_{n=0}^{N-1} x(n).\, e^{\frac{j2\pi n}{N}(k-N)}$$

$$= X(k-N)$$

$$= \left[X\left[k-N\right)\right]^*$$

$$= X^*(-k+N)$$

$$= X^*\left((-k)_N\right)$$

$$= R.H.S$$

7. Parsvel's Theorem:-

If $x(n) \xleftrightarrow{DFT} X(k)$ & $y(n) \xleftrightarrow{DFT} Y(k)$ then,

$$\sum_{n=0}^{N-1} x(n)\, y^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k).\, Y^*(k).$$

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Proof:- Consider R.H.S;

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k).\, Y^*(k)$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \left[\sum_{n=0}^{N-1} x(n).\, e^{-\frac{j2\pi nk}{N}} \right].\, Y^*(k).$$

$$= \sum_{n=0}^{N-1} x(n).\, \left[\frac{1}{N} \sum_{k=0}^{N-1} Y^*(k)\, e^{\frac{-j2\pi nk}{N}} \right]$$

$$= \sum_{n=0}^{N-1} x(n).\, \left[\frac{1}{N} \sum_{k=0}^{N-1} Y(k).\, e^{\frac{-j2\pi nk}{N}} \right]^*$$

$$= \sum_{n=0}^{N-1} x(n) \left[\frac{1}{N} \sum_{k=0}^{N-1} Y(k)\, e^{\frac{j2\pi nk}{N}} \right]^*$$

$$= \sum_{n=0}^{N-1} x(n).\, y^*(n) = L.H.S$$

$\therefore$ L.H.S = R.H.S

8. Time Reversal property:-

If $x(n) \xleftrightarrow{DFT} X(k)$, then

$x((-n))_N \xleftrightarrow{DFT} X((-k))_N$

Proof:- By defination $DFT[x(n)] = \sum_{n=0}^{N-1} x(n).\, e^{\frac{-j2\pi nk}{N}}$

$$DFT[x((-n))_N] = \sum_{n=0}^{N-1} x((-n))_N \cdot e^{\frac{-j2\pi nk}{N}}$$

$$= \sum_{n=0}^{N-1} x(N-n).\, e^{\frac{-j2\pi nk}{N}}$$

Let, $N-n=p$ if $n=0 \Rightarrow p=N$

$n=N-p$, $n=N-1 \Rightarrow N-(N-1)$

$\Rightarrow N-N+1$

$n=1$

$$= \sum_{p=N}^{1} x(p).\, e^{\frac{-j2\pi k(N-p)}{N}}$$

$$= \sum_{p=N}^{1} x(p).\, e^{\frac{-j2\pi kN}{N}} \cdot e^{\frac{-j2\pi kp}{N}}$$

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$$= \sum_{P=N}^{1} x(P) \cdot e^{\frac{j2\pi kP}{N}} \cdot e^{-\frac{j2\pi kN}{N}}$$

$$= \sum_{P=N}^{1} x(P) \cdot e^{\frac{j2\pi kP}{N}}$$

Interchanging the limits.

$$= \sum_{P=1}^{N} x[P] \cdot e^{-\frac{j2\pi P(-k)}{N}} \cdot e^{-\frac{j2\pi N \cdot P}{N}}$$

$$= \sum_{P=1}^{N} x[P] \cdot e^{-\frac{j2\pi P}{N}(N-k)}$$

$$= X(N-k)$$

$$= X((-k))_N$$

$$= R.H.S$$

Convolution using DFT Problems-

(1) Perform the circular convolution of the following sequences. $x(n) = \{1, 1, 2, 1\}$, $h(n) = \{1, 2, 3, 4\}$. using DFT and IDFT method.

(A) we know that $X_3(K) = X_1(K) \cdot X_2(K)$.

$$X_1(K) = \sum_{n=0}^{N-1} x_1(n) e^{-j2\pi nk/N} \quad \text{where } k = 0, 1, \ldots N-1$$

Given $x_1(n) = \{1, 1, 2, 1\}$ and $N = 4$.

$$X_1(0) = \sum_{n=0}^{3} x_1(n) e^{-j2\pi n(0)/4} \Rightarrow \sum_{n=0}^{3} x_1(n)$$

$$X_1(0) \Rightarrow 1+1+2+1 = 5$$

$$X_1(1) = \sum_{n=0}^{3} x_1(n) e^{-j2\pi n(1)/4}$$

$$= \sum_{n=0}^{3} x_1(n) e^{-j\pi n/2} = 1 - j - 2 + j = -1$$

$$X_1(2) = \sum_{n=0}^{3} x_1(n) e^{-j\pi n 2/2}$$

$$= \sum_{n=0}^{3} x_1(n) e^{-j\pi n} = 1 - 1 + 2 - 1 = 1 + 2 - 2 = 1$$

$$X_1(3) = \sum_{n=0}^{3} x_1(n) e^{-j\pi n 3/2} = 1 + j - 2 - j = -1$$

$$\therefore X_1(K) = \{5, -1, 1, -1\}$$

$$X_2(K) = \sum_{n=0}^{N-1} x_2(n) e^{-j2\pi nk/N} \quad \text{where } k = 0, 1, \ldots N-1$$

$$X_2(K) = \sum_{n=0}^{3} x_2(n) \cdot e^{-j\pi nk/2}$$

$$X_2(0) = \sum_{n=0}^{3} x_2(n)\, e^{-j\pi n(0)/2} = \sum_{n=0}^{3} x_2(n) = 1+2+3+4 = 10$$

$$X_2(1) = \sum_{n=0}^{3} x_2(n)\, e^{-j\pi n(1)/2} = \sum_{n=0}^{3} x_2(n)\, e^{-j\pi n/2}$$

$$X_1(1) \Longrightarrow 1+2(-j)+3(-1)+4(j) \Rightarrow -2+j2$$

$$X_2(2) = \sum_{n=0}^{3} x_2(n)\, e^{-j\pi n 2/2} = \sum_{n=0}^{3} x_2(n)\, e^{-j\pi n}$$

$$X_2(2) \Longrightarrow 1+2(-1)+3(1)+4(-1) = -2-4+4 = -2$$

$$X_2(3) = \sum_{n=0}^{3} x_2(n)\, e^{-j\pi n\, 3/2} = 1+2j+3(-1)+4(-j) = -2-2j$$

$$\therefore X_2(K) = \{10, -2+2j, -2, -2-2j\}$$

$$X_3(K) = X_1(K) \cdot X_2(K) = \{50, 2-2j, -2, 2+2j\}$$

Convolution using 2 DFT.

$$x(n) = \{1, 1, 2, 1\} \quad , h(n) = \{1, 2, 3, 4\}$$

$$x_3(n) = \frac{1}{N} \sum_{K=0}^{N-1} X_3(K)\, e^{j2\pi nk/N} \qquad \text{where } n = 0, 1, 2, \ldots N-1$$

$$\therefore x_3(n) = \frac{1}{4} \sum_{K=0}^{3} X_3(K)\, e^{j\frac{2\pi nk}{4}}$$

$$x_3(n) = \frac{1}{4} \sum_{K=0}^{3} X_3(K)\, e^{j\frac{\pi nk}{2}}$$

$$x_3(0) = \frac{1}{4} \sum_{K=0}^{3} X_3(K)\, e^{j\frac{\pi(0)k}{2}}$$

$$= \frac{1}{4}\left(50 + 2 - 2j - 2 + 2 + 2j\right) = \frac{52}{4} = 13$$

$$x_3(1) = \frac{1}{4}\left[\sum_{k=0}^{3} x_3(k)\, e^{j\frac{k\pi(1)}{2}}\right]$$

$$= \frac{1}{4}\left[50 + (2-2j)(+j) + (-2)(-1) + (2+2j)(-j)\right] = 14$$

$$x_3(2) = \frac{1}{4}\left[\sum_{k=0}^{3} x_3(k) e^{j\frac{k\pi(2)}{2}}\right]$$

$$= \frac{1}{4}\left[50 + (2-2j)(-1) + (-2)(1) + (2+2j)(-1)\right] = 11$$

$$x_3(3) = \frac{1}{4}\left[\sum_{k=0}^{3} x_3(k) e^{j3\pi k/2}\right]$$

$$= \frac{1}{4}\left[50 + (2-j2)(-j) + (-2)(-1) + (2+2j)(j)\right] = 12$$

$$\therefore x_3(n) = \{13, 14, 11, 12\}$$

(2) Compute the DFT of a sequence $(-1)^n$ for $N=4$.

Sol:- Given $x(n) = (-1)^n$.

$x(0) = 1, x(1) = -1, x(2) = 1, x(3) = -1$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} \qquad k = 0, 1, \ldots N-1.$$

$$= \sum_{n=0}^{3} x(n) e^{-j2\pi nk/N} = \sum_{n=0}^{3} x(n) \cdot e^{-j\pi nk/2}.$$

$$X(0) = \sum_{n=0}^{3} x(n) = x(0) + x(1) + x(2) + x(3) = 1 - 1 + 1 - 1 = 0$$

$$X(1) = \sum_{n=0}^{3} x(n) e^{-j\pi n/2} = 1 - e^{-j\pi/2} + e^{-j\pi} - e^{-j3\pi/2}$$

$$= 1 + j - 1 - j = 0$$

$$X(2) = \sum_{n=0}^{3} x(n) e^{-j\pi n} = 1 - e^{-j\pi} + e^{-j2\pi} - e^{-j3\pi}$$

$$= 1 + 1 + 1 + 1 = 4$$

$$X(3) = \sum_{n=0}^{3} x(n) e^{-j3\pi n/2} = 1 - e^{-j3\pi/2} + e^{-j3\pi} + e^{-j9\pi/2}$$

$$= 1 - j - 1 + j$$

$$= 0$$

$$\therefore X(k) = \{0, 0, 4, 0\}$$

DFT Problems:-

① Find the DFT of a sequence $x(n) = \{1, 1, 0, 0\}$ and find the IDFT of $Y(k) = \{1, 0, 1, 0\}$.

(A) Let us assume $N = L = 4$. DFT of $x(n)$ is $X(k)$

i.e, we have $X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N}$

where $k = 0, 1, \ldots N-1$.

$$X(k) = \sum_{n=0}^{3} x(n) \cdot e^{-j2\pi nk/4}$$

$$\therefore X(k) = \sum_{n=0}^{3} x(n) e^{-j\pi nk/2}$$

for $k = 0$

$$X(0) = \sum_{n=0}^{3} x(n) \cdot e^{-j\pi n(0)/2}$$

$$= \sum_{n=0}^{3} x(n) \cdot e^{0} \Rightarrow \sum_{n=0}^{3} x(n).$$

$$X(0) = 1 + 1 + 0 + 0 = 2$$

for $k = 1$

$$X(1) = \sum_{n=0}^{3} x(n) e^{-j\pi n(1)/2}$$

$$X(1) = \sum_{n=0}^{3} x(n) e^{-j\pi n/2}$$

$$= x(0)\cdot e^{0} + x(1)\cdot e^{-j\pi(1)/2} + x(2)\cdot e^{-j\pi(2)/2} + x(3)\cdot e^{-j\pi(3)/2}$$

$$= 1 + \cos\pi/2 - j\sin\pi/2$$

$$X(1) = 1 - j$$

$$X(2) = \sum_{n=0}^{3} x(n) e^{-j n\pi \frac{2}{2}}$$

$$= \sum_{n=0}^{3} x(0)\cdot e^{-j\pi 0/2} + x(1)e^{-j\pi(1)/2} + x(2)e^{-j\pi(2)/2} + x(3)e^{-j\pi 3/2}$$

$$= x(0)\cdot e^{0} + \cos\pi - j\sin\pi \Rightarrow 1 - 1 = 0$$

$$X(3) = \sum_{n=0}^{3} x(n) e^{-j3n\pi/2}$$

$$= \sum_{n=0}^{3} x(0) e^{\frac{-j\pi 3(0)}{2}} + x(1)e^{\frac{-j\pi 3(1)}{2}} + x(2)e^{\frac{-j3\pi(2)}{2}} + x(3)e^{\frac{-j3\pi(3)}{2}}$$

$$= 1 + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} \Rightarrow 1 + j$$

$$\therefore X(K) = \{2, 1-j, 0, 1+j\}$$

IDFT :-

$$x(n) = \frac{1}{N}\sum_{k=0} X(k)\cdot e^{\frac{j2\pi nk}{N}}$$

where $n = 0, 1, \ldots N-1$

from the given Eq.

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k) e^{\frac{j2\pi nk}{N}}$$

where $n = 0, 1, \ldots N-1$.

$$y(n) = \frac{1}{4}\sum_{k=0}^{3} Y(k) e^{\frac{j2\pi nk}{4}}$$

$n = 0, 1, 2, 3$

$$y(0) = \frac{1}{4}\sum_{k=0}^{3} Y(k).e^{\frac{j2\pi n(0)}{2}}$$

$$= \frac{1}{4}\sum_{k=0}^{3} Y(k)$$

$$= \frac{1}{4}\left[Y(0)+Y(1)+Y(2)+Y(3)\right] = \frac{1}{4}[1+0+1+0]$$

$$= \frac{1}{4}(2)$$

$$y(0) = \frac{2}{2} = 0.5$$

$$y(1) = \frac{1}{4}\sum_{k=0}^{3} Y(k)\, e^{\frac{j\pi K.(1)}{2}}$$

$$= \frac{1}{4}\left[Y(0).e^{0}+Y(1)e^{j\pi/2}+Y(2)e^{j\pi}+Y(3)e^{j3\pi/2}\right]$$

$$= \frac{1}{4}\left[1+0+\cos\pi + j\sin\pi + 0\right].$$

$$= \frac{1}{4}\left[1+0-1+0\right] = 0$$

$$y(2) = \frac{1}{4}\left[Y(0)+Y(1)e^{j\pi}+Y(2)e^{j2\pi}+Y(3)e^{j3\pi}\right]$$

$$= \frac{1}{4}\left[1+0+\cos 2\pi + j\,2\sin\pi + 0\right]$$

$$= \frac{1}{4}\left[1+0+1+0\right] 2\ 0.5$$

$$y(3) = \frac{1}{4}\left[Y(0)+Y(1)e^{j3\pi/2}+Y(2)e^{j3\pi}+Y(3)e^{j3\pi/2}\right]$$

$$= \frac{1}{4}\left[1+0+\cos 3\pi + j\sin 3\pi + 0\right]$$

$$= \frac{1}{4}\left[1+0+(-1)+0\right] = 0$$

$$\therefore y(n) = \{0.5, 0, 0.5, 0\}$$

(2) Find the DFT of a sequence. $x(n) = 1$ for $0 \le n \le 2$
$= 0$ otherwise
for i) $N=4$, ii) $N=8$. Plot $|X(k)|$ and $\angle X(k)$.

Sol:- Given $L=3$, For $N=4$, the periodic extension of $x(n)$ can be obtained by adding one zero (i.e. $N-L$ zeros). for $N=4$ and adding four zeros for $N=8$.

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi nk}{N}} \quad \text{where } k = 0, 1, \ldots N-1$$

$\therefore x(n) = \{1, 1, 1, 0\}$

$x(0)=1$, $x(1)=1$, $x(2)=1$, $x(3)=0$..

for $N=4$

$$X(k) = \sum_{n=0}^{3} x(n) e^{-j\frac{\pi nk}{2}}, \quad k = 0, 1, 2, 3.$$

for $k=0$

$$X(0) = \sum_{n=0}^{3} x(n)\, e^{-j\frac{\pi n(0)}{2}}$$

$$= x(0)+x(1)+x(2)+x(3). \Rightarrow 1+1+1+0$$

$$X(0) = 3$$

$\therefore |X(0)| = 3$, $\angle X(0) = 0$.

for $k=1$

$$X(1) = \sum_{n=0}^{3} x(n) e^{-j\frac{\pi n(1)}{2}}$$

$$= x(0)+x(1)e^{-j\pi(1)/2} + x(2)\cdot e^{-j\pi\, 2/2} + x(3)e^{-j\pi\, 3/2}$$

$$= x(0)+(1)\cdot e^{-j\pi/2} + (1).e^{-j\pi} + 0.$$

$$= 1 + \cos\pi/2 - j\sin\pi/2 + \cos\pi - j\sin\pi$$

$$X(1) = 1 - j - 1 = -j$$

$\therefore |X(1)| = 1, \quad \angle X(1) = -\pi/2.$

for $K=2$, $X(2) = \sum_{n=0}^{3} x(n) e^{-j\pi n}$

$= x(0) + x(1) e^{-j\pi} + x(2) e^{-j2\pi} + x(3) e^{-j3\pi}$

$= 1 + (1)\cos\pi - j\sin\pi + (1)\cos 2\pi - j\sin 2\pi + 0$

$X(2) = 1 - 1 + 1 = 2 - 1 = 1$

$\therefore |X(2)| = 1, \quad \angle X(2) = 0.$

for $K=3$, $X(3) = \sum_{n=0}^{3} x(n) e^{-j3\pi n/2}$

$= x(0) + x(1) e^{-j3\pi/2} + x(2) \cdot e^{-j3\pi} + x(3) e^{-j9\pi/2}$

$= 1 + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} + \cos 3\pi - j\sin 3\pi + 0$

$= 1 + j - 1 = j$

$\therefore |X(3)| = 1, \quad \angle X(3) = \pi/2$

$\therefore |X(k)| = \{3, 1, 1, 1\}$

$\angle X(k) = \{0, \pi/2, 0, \pi/2\}$

The plot of $|X(k)|$ and $\angle X(k)$ for $N=4$ is shown in below figure

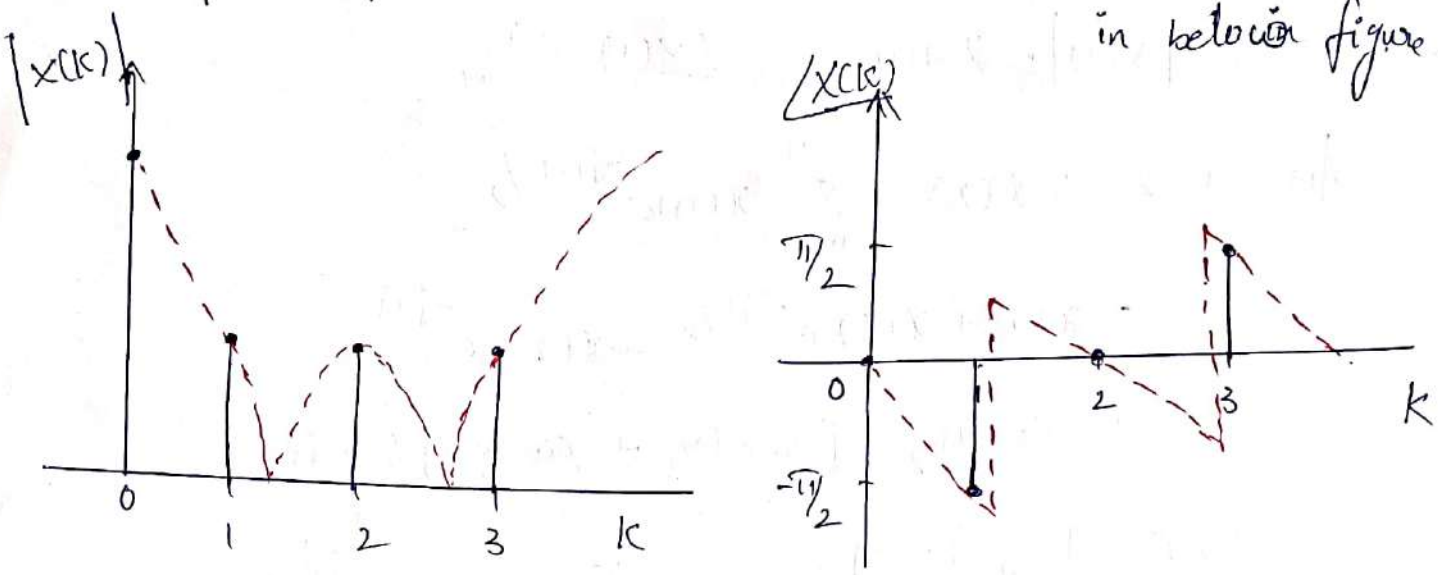

fig: frequency response of $x(n)$ for $N=4$.

⇒ for $N=8$ the periodic extension of $x(n)$ can be obtained by adding five zeros (i.e. N-L zeros)

$x(n)=\{1,1,1,0,0,0,0,0\}$

$x(0)=x(1)=x(2)=1$ and $x(n)=0$ for $3\le n\le 7$

we have $X(k)=\sum_{n=0}^{N-1} x(n)\cdot e^{-j2\pi nk/N}$

for $N=8$ $\quad X(k)=\sum_{n=0}^{7} x(n)e^{-j\pi nk/4}$ where $k=0,1\cdots 7$

for $k=0$, $\quad X(0)=\sum_{n=0}^{7} x(n)$.

$X(0)=1+1+1+0+0+0+0+0=3.$

$\therefore |X(k)|=3$, $\quad \angle X(k)=0$

for $k=1$, $\quad X(1)=\sum_{n=0}^{7} x(n)e^{\frac{-j\pi n(1)}{4}}$

$=x(0)+x(1)e^{-j\pi(1)/4}+x(2)e^{-j\pi/2}$

$= 1 + 0.707 - j0.707 + 0 - j = 1.707 - j1.707$

$\therefore |X(1)| = 2.414, \quad \angle X(1) = -\pi/4.$

for $k=2$; $X(2) = \sum_{n=0}^{7} x(n) e^{-j\pi n/2}$

$= x(0) + x(1) e^{-j\pi/2} + x(2) e^{-j\pi}$

$= 1 + \cos \pi/2 - j \sin \pi/2 + \cos \pi - j \sin \pi$

$X(2) = 1 - j + 1 = -j$

$\therefore |X_2(2)| = 1 \quad \angle X(2) = -\pi/2$

for $k=3$, $X(3) = \sum_{n=0}^{7} x(n) e^{-j3\pi n/4}$

$= x(0) + x(1) e^{-j3\pi/4} + x(2) e^{-j3\pi/2}$

$= 1 + \cos \frac{3\pi}{4} - j \sin \frac{3\pi}{4} + \cos \frac{3\pi}{2} - j \sin \frac{3\pi}{2}$

$= 1 - 0.707 - j0.707 + j$

$= 0.293 + j0.293$

$\therefore |X(3)| = 0.414, \quad \angle X(3) = \frac{\pi}{4}$

for $k=4$

$X(4) = \sum_{n=0}^{7} x(n) e^{-j\pi n}$

$= x(0) + x(1) e^{-j\pi} + x(2) e^{-j2\pi}$

$= 1 + \cos \pi - j \sin \pi + \cos 2\pi - j \sin 2\pi$

$= 1 - 1 + 1 = 1$

$\therefore |X(4)| = 1, \quad \angle X(4) = 0$

for $K=5$, $X(5)=\sum_{n=0}^{7} x(n)e^{-j5\pi n/4}$.

$=x(0)+x(1)e^{-j5\pi/4}+x(2)e^{-j3\pi/2}$

$=1+\cos\frac{5\pi}{4}-j\sin\frac{5\pi}{4}+\cos\frac{3\pi}{2}-j\sin\frac{3\pi}{2}$.

$=1-0.707+j0.707-j$

$=0.293-j0.707$.

$\therefore |X(5)|=0.414$, $\angle X(5)=\pi/4$.

for $k=6$, $x(6)=\sum_{n=0}^{7} x(n)e^{-j3\pi/2}+x(2).e^{-j3\pi}$

$=1+\cos 3\pi/2-j\sin 3\pi/2+\cos 3\pi-j\sin 3\pi$

$=1+j-1=j$

$\therefore |X(6)|=1$, $\angle X(6)=\pi/2$

for $k=7$, $x(7)=\sum_{n=0}^{7} x(n)e^{-j7\pi n/4}$

$=x(0)+x(1)e^{-j7\pi/4}+x(2)e^{-j7\pi/2}$

$=1+\cos\frac{7\pi}{4}-j\sin\frac{7\pi}{4}+\cos\frac{7\pi}{2}-j\sin\frac{7\pi}{2}$.

$=1+0.707+j0.707+j$

$=1.707+j0.707$.

$|X(7)|=2.414$, $\angle X(7)=\pi/4$.

$|X(K)|=\{3, 2.414, 1, 0.414, 1, 0.414, 1, 2.414\}$

$\angle X(K)=\{0, -\pi/4, -\pi/2, \pi/4, 0, -\pi/4, \pi/2, \pi/4\}$

The plot of $|X(k)|$ and $\angle X(k)$ Vs k for $N=8$. is

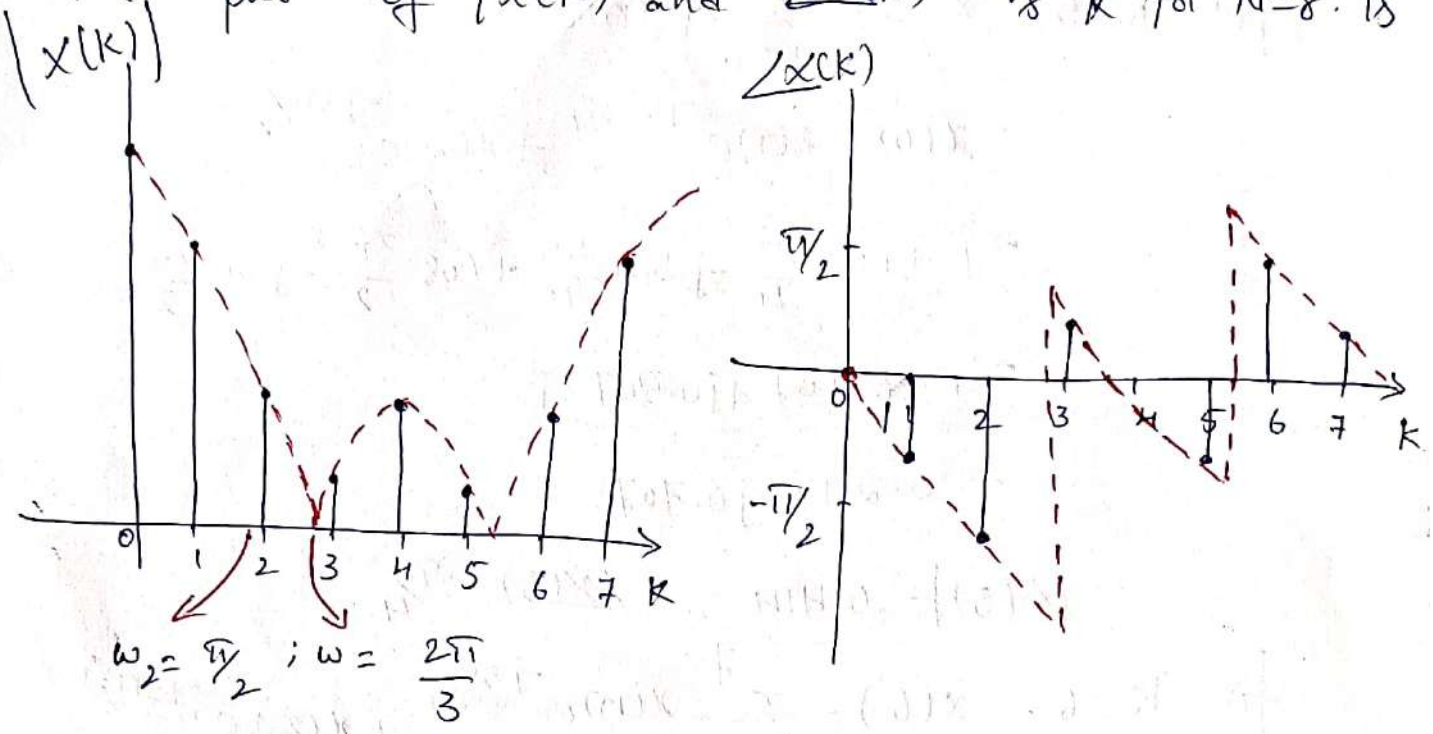

fig:- Frequency response of $x(n)$ for $N=8$.

(3) Determine the 8-point DFT of the sequence.

$x(n) = \{1, 1, 1, 1, 1, 1, 0, 0\}$.

Sol:- we know that $X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi nk}{N}}$

where $k = 0, 1, \ldots N-1$.

for $N=8$.

$$X(k) = \sum_{n=0}^{7} x(n) e^{-j\pi nk/4}$$

$k = 0, 1, 2, 3, 4, 5, 6, 7$.

for $k=0$:

$$X(0) = \sum_{n=0}^{7} x(n) e^{0}.$$

$$X(0) = x(0)+x(1)+x(2)+x(3)+x(4)+x(5)+x(6)+x(7)$$

$$= 1+1+1+1+1+1+0+0 = 6.$$

$$X(0) = 6$$

for $k=1$;

$$X(1)=\sum_{n=0}^{7} x(n)e^{-j\pi n/4}$$

$$=x(0)+x(1)e^{-j\pi/4}+x(2)e^{-j\pi/2}+x(3)e^{-j3\pi/4}+x(4)e^{-j\pi}$$
$$+x(5)e^{-j5\pi/4}+x(6)e^{-j3\pi/2}+x(7)e^{-j7\pi/4}.$$

$$=1+0.707-j0.707-j-0.707-j0.707-1-0.707+j0.707$$

$$X(1)=-0.707-j0.707$$

for $k=2$;

$$X(2)=\sum_{n=0}^{7} x(n)e^{-j\pi n/2}$$

$$=x(0)+x(1)e^{-j\pi/2}+x(2)e^{-j\pi}+x(3)e^{-j3\pi/2}+x(4)$$
$$e^{-j2\pi/2}+x(5)e^{-j5\pi/2}+x(6)e^{-j3\pi}+x(7)e^{-j7\pi/2}$$

$$X(1)=1-j-1+j+1-j \Rightarrow 1-j$$

for $k=3$;

$$X(3)=\sum_{n=0}^{7} x(n)e^{-j3\pi n/4}.$$

$$=x(0)+x(1)e^{-j3\pi/4}+x(2)e^{-j3\pi/2}+x(3)e^{-j9\pi/4}+$$
$$x(4)e^{-j3\pi}+x(5)e^{-j15\pi/4}+x(6)e^{-j9\pi/2}+x(7)e^{-j21\pi/4}.$$

$$=1-0.707-j0.707+j+0.707-j0.707-1+0.707+j0.707.$$

$$X(3)=0.707+j0.293$$

for $k=4$;

$$X(4)=\sum_{n=0}^{7} x(n)e^{-j\pi n}$$

$$=x(0)+x(1)e^{-j\pi}+x(2)e^{-j2\pi}+x(3)e^{-j3\pi}+x(4)e^{-j4\pi}+$$
$$x(5)e^{-j5\pi}+x(6)e^{-j6\pi}+x(7)e^{-j7\pi}$$

$X(4) = 1-1+1-1+1-1 = 0$

for $k=5$;

$$X(5) = \sum_{n=0}^{7} x(n) e^{-j5\pi n/4}$$

$$= x(0) + x(1)e^{-j5\pi/4} + x(2)e^{-j5\pi/2} + x(3)e^{-j15\pi/4} + x(4)e^{-j5\pi}$$
$$+ x(5)e^{-j25\pi/4} + x(6)e^{-j15\pi/2} + x(7)e^{-j35\pi/4}.$$

$$= 1 - 0.707 + j0.707 - j + 0.707 + j0.707 - 1 + 0.707 - j0.707$$

$X(5) = 0.707 - j0.293.$

for $k=6$;

$$X(6) = \sum_{n=0}^{7} x(n) e^{-j3\pi n/2}$$

$$= x(0) + x(1)e^{-j3\pi/2} + x(2)e^{-j3\pi} + x(3)e^{-j9\pi/2} + x(4)e^{-j6\pi}$$
$$+ x(5)e^{-j15\pi/2} + x(6)e^{-j9\pi} + x(7)e^{-j21\pi/2}.$$

$$= 1 + j - 1 - j + 1 + j$$

$X(5) = 1 + j$

for $k=7$;

$$X(7) = \sum_{n=0}^{7} x(n) e^{-j7\pi n/4}$$

$$= x(0) + x(1)e^{-j7\pi/4} + x(2)e^{-j7\pi/2} + x(3)e^{-j21\pi/4} +$$
$$x(4)e^{-j7\pi} + x(5)e^{-j35\pi/4} + x(6)e^{-j21\pi/2} + x(7)e^{-j49\pi/4}.$$

$$= 1 + 0.707 + j0.707 + j - 0.707 + j0.707 - 1 - 0.707 - j0.707$$

$X(7) = -0.707 + j0.707.$

$\therefore X(K) = \{6, -0.707 - j0.707, 1-j, 0.707 + j0.293, 0, 0.707 - j0.293,$
$1+j, -0.707 + j1.707\}$

(8) Find 2DFT of the sequence. $X(k) = \{5, 0, 1-j, 0, 1, 0, 1+j, 0\}$.

Sol: we have $x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) \cdot e^{j2\pi kn/N}$ where $n = 0, 1, \ldots N-1$.

for $N = 8$.

$$x(n) = \frac{1}{8} \sum_{K=0}^{7} X(k)\, e^{\frac{j\pi nk}{4}}$$ $n = 0, 1, \ldots 7$.

for $n = 0$:

$$x(0) = \frac{1}{8}\left[\sum_{k=0}^{7} X(k)\right] = \frac{1}{8}\left[5+0+1-j+0+1+0+1+j+0\right]$$

$$x(0) = \frac{8}{8} = 1 \qquad = 1$$

$$x(1) = \frac{1}{8}\left[\sum_{k=0}^{7} X(k)\, e^{j\pi k/4}\right]$$

$$= \frac{1}{8}\left[5 + (1-j)(j) + 1(-1) + (1+j)(-j)\right]$$

$$= \frac{1}{8}[6] = 0.75$$

$$x(2) = \frac{1}{8}\left[\sum_{k=0}^{7} X(k) e^{j\pi k/2}\right]$$

$$= \frac{1}{8}\left[5 + (1-j)(-1) + (1)1 + (1+j)(-1)\right]$$

$$= \frac{1}{8}(4) = 0.5$$

$$x(3) = \frac{1}{8}\left[\sum_{k=0}^{7} X(k) e^{j3\pi k/4}\right]$$

$$= \frac{1}{8}\left[5 + (1-j)(j) + 1(-1) + (1+j)(j)\right]$$

$$= \frac{1}{8}(2) = 0.25$$

$$X(4)=\frac{1}{8}\left[\sum_{k=0}^{7} x(k)e^{j\pi k}\right]$$

$$=\frac{1}{8}\left[5+(1-j)(1)+1(1)+(1+j)(1)\right]=1$$

$$X(5)=\frac{1}{8}\left[\sum_{k=0}^{7} x(k)e^{j5\pi k/4}\right]$$

$$=\frac{1}{8}\left[5+(1-j)(j)+1(-1)+(1+j)(1)\right]=\frac{1}{8}(6)=0.75$$

$$X(6)=\frac{1}{8}\left[\sum_{k=0}^{7} x(k)e^{j3\pi k/2}\right]$$

$$=\frac{1}{8}\left[5+(1-j)(-1)+1(1)+(1+j)(-1)\right]=\frac{1}{8}(4)=0.5$$

$$X(7)=\frac{1}{8}\left[\sum_{k=0}^{7} x(k)e^{j7\pi k/4}\right]$$

$$=\frac{1}{8}\left[5+(1-j)(-j)+1(-1)+(1+j)(j)\right]=\frac{1}{8}(2)=0.25$$

$\therefore x(n)=\{1, 0.75, 0.5, 0.25, 1, 0.75, 0.5, 0.25\}$.

(5) Compute 4-point DFT of the sequence

$x(n)=\{0, 2, 4, 6\}$

Ans: $12, -4+j4, -4, -4-4j$.

Relationship of DFT to Other Transforms:-

Relationship to the Fourier Transform:-

The Fourier Transform $X(e^{j\omega})$ of a finite duration sequence $x(n)$ having length N is

$$X(e^{j\omega})=\sum_{n=0}^{N-1} x(n)e^{-j\omega n} \longrightarrow (1)$$

where $X(e^{j\omega})$ is a continuous function of ω.

The discrete Fourier Transform of $x(n)$ is given by

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N}, \quad K=0,1,2,\ldots N-1. \longrightarrow (2)$$

Comparing eqn (1) & (2) we find that the DFT of $x(n)$ is a sampled version of the fourier transform of the sequence and is given by.

$$X(K) = X(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{N}} \quad K=0,1,\ldots N-1$$

Relationship to the Z-Transform:-

Let us consider a sequence $x(n)$ of finite duration N with Z-Transform.

$$X(Z) = \sum_{n=0}^{N-1} x(n) z^{-n}. \longrightarrow (3)$$

we have $x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{\frac{j2\pi kn}{N}} \longrightarrow (4)$

Substituting eqn (4) in eqn (3) we get

$$X(Z) = \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{j2\pi kn/N} \right] z^{-n}$$

$$= \frac{1}{N} \sum_{K=0}^{N-1} X(K) \sum_{n=0}^{N-1} \left(e^{j2\pi nk/N} z^{-1} \right)^n$$

$$= \frac{1-z^{-N}}{N} \sum_{K=0}^{N-1} \frac{X(K)}{1 - e^{j2\pi K/N} z^{-1}} \longrightarrow (5)$$

* Consider the length-8 sequence defined for $0 \le n \le 8$, $x(n) = \{1, 2, -3, 0, 1, -1, 4, 2\}$ with a 8-point DFT. Evaluate the following functions of $X(K)$ without computing DFT.

(a) $X(0)$

We know that $X(K) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N}$

$\therefore X(K) = \sum_{n=0}^{7} x(n) e^{-j\pi nk/4}$

$\therefore X(0) = \sum_{n=0}^{7} x(n) e^{0}$.

$= x(0) + x(1) + x(2) + x(3) + x(4) + x(5) + x(6) + x(7)$

$= 1 + 2 - 3 + 0 + 1 - 1 + 4 + 2$

$X(0) = 10 - 4 \Rightarrow 6$

(b) $X(4)$.

W.K.T. $X(4) = \sum_{n=0}^{7} x(n) e^{-j\pi n 4/4}$

$X(4) = \sum_{n=0}^{7} x(n) e^{-j\pi n}$

$\Rightarrow x(0) e^{0} + x(1) e^{-j\pi(1)} + x(2) e^{-j\pi(2)} + x(3)$

$= \sum_{n=0}^{7} x(4) e^{j\pi 0}$

$= 0 \cdot e^{0}$

$\therefore X(4) = 0$

(c) $\sum_{K=0}^{7} x(K)$.

from $x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{\frac{j2\pi nk}{N}}$

$= \frac{1}{N} \cdot \sum_{n=0}^{7} N x(0)$. $\sum_{K=0}^{N-1} X(K) = N x(0)$

$= 8$.

(d) $\sum_{K=0}^{7} e^{\frac{-j3\pi K}{4}} X(K)$

$DFT\left[x(n-3)_8\right] = e^{\frac{-j3\pi k}{4}} X(K)$

Take IDFT of $e^{\frac{-j3\pi K}{4}} X(K)$ and substitute $n=0$

$\frac{1}{N} \sum_{K=0}^{7} e^{\frac{-j3\pi K}{4}} X(K) = x((0-3))_8$

$\Rightarrow \sum_{K=0}^{7} e^{\frac{-j3\pi K}{4}} X(K) = N x((0-3))_8 = N x(5)$.

$= 8.(1) = -8$

(e) $\sum_{K=0}^{7} |X(K)|^2$

From the Parseval's Theorem

$$\sum_{K=0}^{7} |X(K)|^2 = N \sum_{n=0}^{7} |x(n)|^2.$$

---

* Determine IDFT of $X(K) = \{12, -1.5+j2.598, -1.5+j0.866, 0, -1.5-j0.866, -1.5-j2.598\}$

Sol:- we know $x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) . e^{\frac{j2\pi nk}{N}}$

$N = 6$.

$n = 0, 1, \ldots N-1$

$$x(n) = \frac{1}{6}\sum_{k=0}^{5} X(k)\cdot e^{j2\pi kn/6} \Rightarrow \frac{1}{6}\sum_{k=0}^{5} X(k)\, e^{j\frac{2\pi kn}{3}}$$

$$x(0) = \frac{1}{6}[X(0)+X(1)+X(2)+X(3)+X(4)+X(5)]$$

$$= \frac{1}{6}[12-1.5+j2.598-1.5+j0.866+0-1.5-j0.866-1.5-j2.598] = 0$$

$$x(0) = \frac{6}{6} = 1$$

$$x(1) = \frac{1}{6}\left[x(0)+X(1)e^{\frac{j\pi(1)(1)}{3}}+X(2)e^{\frac{j\pi 2(1)}{3}}+X(3)e^{\frac{j\pi 3(1)}{3}}+X(4)e^{\frac{j\pi 4(1)}{3}}+X(5)\cdot e^{\frac{j\pi 5(1)}{3}}\right]$$

$$x(1) = \frac{1}{6}\left[12+(-1.5+j2.598)e^{j\pi/3}+(-1.5+j0.866)e^{j2\pi/3}+0(-1)+(-1.5-j0.866)e^{j4\pi/3}+(-1.5-j2.598)e^{j5\pi/3}\right]$$

$$= \frac{1}{6}\left[12+(-1.5+j2.598)(0.5+j0.866)+(-1.5+j0.866)(-0.5+j0.866)+0+(-1.5-j0.866)(-0.5-j0.866)+(-1.5-j2.598)(0.5-j0.866)\right]$$

$$= 1.$$

$$x(2) = \frac{1}{6}\left[x(0)+X(1)e^{j2\pi/3}+X(2)e^{j4\pi/3}+X(3)e^{j2\pi}+X(4)e^{j8\pi/3}+X(5)e^{j10\pi/3}\right]$$

$$= \frac{1}{6}\left[12+(-1.5+j2.598)(-0.5+j0.866)+(-1.5+j0.866)(-0.5+j0.866)+0+(-1.5-j0.866)(-0.5+j0.866)+(-1.5-j2.598)(-0.5-j0.866)\right]$$

$$x(2) = \frac{1}{6}(12) = 2.$$

$$x(3) = \frac{1}{6}\left[x(0)+X(1)e^{j\pi}+X(2)e^{j2\pi}+X(3)e^{j3\pi}+X(4)e^{j4\pi}+X(5)e^{j5\pi}\right]$$

$$= \frac{1}{6}\left[12+(-1.5+j2.598)(-1)+(-1.5+j0.866)(1)+0+(-1.5-j0.866)(1)+(-1.5-j2.598)(-1)\right]$$

$x(3) = \frac{1}{6}(12) = 2$

$$x(4) = \frac{1}{6}\left[x(0) + x(1)e^{j4\pi/3} + x(2)e^{j8\pi/3} + x(3)e^{j4\pi} + x(4)e^{j16\pi/3} + x(5)e^{j\frac{20\pi}{3}}\right]$$

$$= \frac{1}{6}\left[12 + (1.5 + j2.598)(-0.5 + j0.866) + (-1.5 + j0.866)(-0.5 + j0.866) + 0 + (-1.5 - j0.866)(-0.5 - j0.866) + (-1.5 - j2.598)(-0.5 + j0.866)\right]$$

$x(4) = \frac{1}{6}[18] = 3$

$$x(5) = \frac{1}{6}\left[x(0) + x(1)e^{j5\pi/3} + x(2)e^{j10\pi/3} + x(3)e^{j5\pi} + x(4)e^{j20\pi/3} + x(5)e^{j25\pi/3}\right]$$

$$= \frac{1}{6}\left[12 + (-1.5 + j2.598)(0.5 - j0.866) + (-1.5 + j0.866)(-0.5 - j0.866) + 0 + (-1.5 - j0.866)(-0.5 + j0.866) + (-1.5 - j2.598)(0.5 + j0.866)\right]$$

$x(5) = \frac{1}{6}(18) = 3$

$\therefore x(n) = \{1, 1, 2, 2, 3, 3\}$

Fast Fourier Transforms (FFT):-

1. The fast fourier transforms is a method (or) algorithm for computing the discrete fourier transforms with reduced number of calculations.

2. The combinational efficiency is achieved if we adopt a divide & conquer. This approach is based on the decomposition of N-point DFT into the successfully smaller DFT's.

3. This basic approach leads to a family of an efficient computational algorithms known as "Fast Fourier Transforms" algorithms.

4. There are two types of FFT algorithms. They are,
   1. DIT-FFT algorithm (Decimation In Time-FFT).
   2. DIF-FFT algorithm (Decimation In frequency-FFT).

N-point DFT Calculations:-

* The direct computation of N-point DFT requires
  a. N^2 complex multiplications &
  b. $N(N-1)$ Complex addition.

Radix-2 DIT-FFT Calculations:-

1. Number of complex multiplications required are $\boxed{\frac{N}{2}\log_2 N}$.

2. Number of complex additions required are $\boxed{N\log_2 N}$.

Radix-2 DIT-FFT Algorithms:-

1. For performing Radix-2 FFT the value of r should be such that $\boxed{N=2^m}$ Where "r" is the radix & "m" represents the stages of computation

2. Let us consider 8-point DFT which is to be computed using radix-2 DIT-FFT algorithm.

3. In DIT-algorithm the time domain sequence x(n) is decimated and smaller DFT's are computed and they are combined to get N-point DFT.

4. Let x(n) be the 8-point sequence given below.

$x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)\}$.

Bit - Reversal Order :-

Index	Binary Representation	Bit - Reversal Representation	Bit-Reversal Index	
0	000	000	0	Group-I
1	001	100	4	
2	010	010	2	Group-II
3	011	110	6	
4	100	001	1	Group-III
5	101	101	5	
6	110	011	3	Group-IV
7	111	111	7	

Butterfly Diagram :-

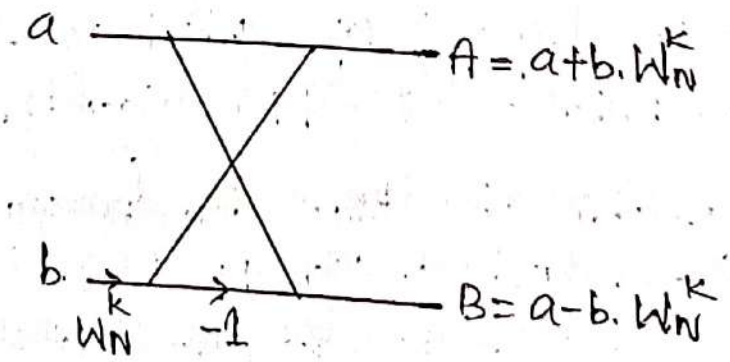

Fig:- Butterfly Diagram of Radix-2 DIT-FFT Algorithm

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Radix-2 DIT-FFT Algorithm:-

1. The number of input samples "N" = 2^M
   Where, M = an integer.
2. The input sequence is shuffled through bit reversal.
3. The number of stages in flow graph is $M = \log_2 N$.
4. Each stage consists of $\frac{N}{2}$ butterflies.
5. Input and output for each butterflies are seperated by 2^{m-1} samples; where m represents the
6. stage index i.e., for first stage m=1 for second stage n=2 & so on.
6. The number of complex multiplication is given by $\frac{N}{2} \log_2 N$.
7. The number of complex additions is given by $N \log_2 N$.
8. The twiddle factor exponents are a function of stage index m and is given by,

$$k = \frac{N.t}{2^m}, \quad t = 0, 1, \ldots 2^{m-1}$$

9. The number of sets (or) section of butterflies in each stage is given by the formula 2^{M-m}
10. The exponent repeat factor (ERF) which is the number of times the exponent sequence associates with m is repeated is given by 2^{M-m}.

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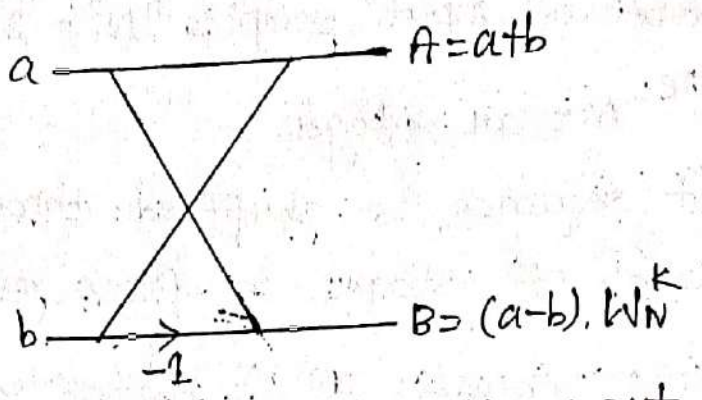
Radix-2 DIF-FFT Algorithm:-



* In DIF-FFT algorithm, at the input side takes the normal order of the input samples and the output sides take the samples in bit reversal order.

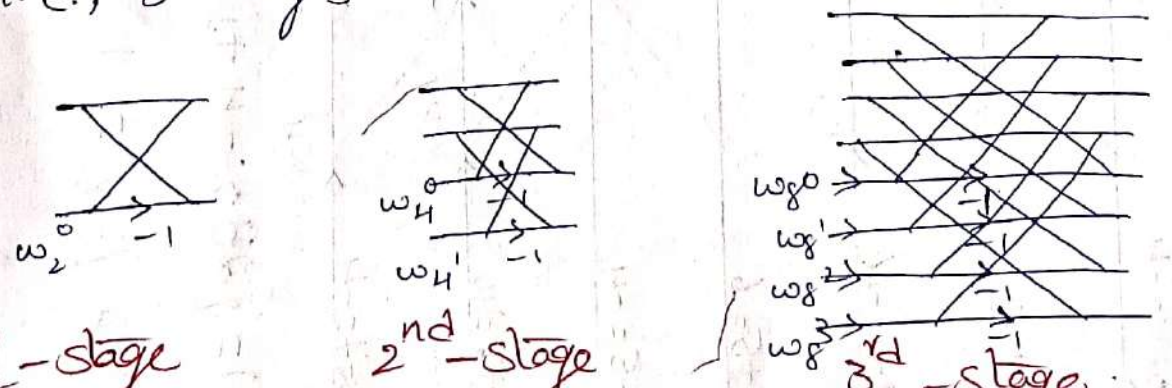
Decimation in Time (DIT-FFT) Algorithm:-

Time domain signal can be converted into frequency domain signal by using DIT-FFT Algorithm.

Eg:- $x(n) = \{0, 1, 2, 3, 4, 5, 6, 7\}$

Natural form	Binary	Bit reversed form	
$x(0)$	000	000	$x(0)$
$x(1)$	001	100	$x(4)$
$x(2)$	010	010	$x(2)$
$x(3)$	011	110	$x(6)$
$x(4)$	100	001	$x(1)$
$x(5)$	101	101	$x(5)$
$x(6)$	110	011	$x(3)$
$x(7)$	111	111	$x(7)$

There are 8 samples in the sequence so $8 = 2^3$ i.e., 3-stages.

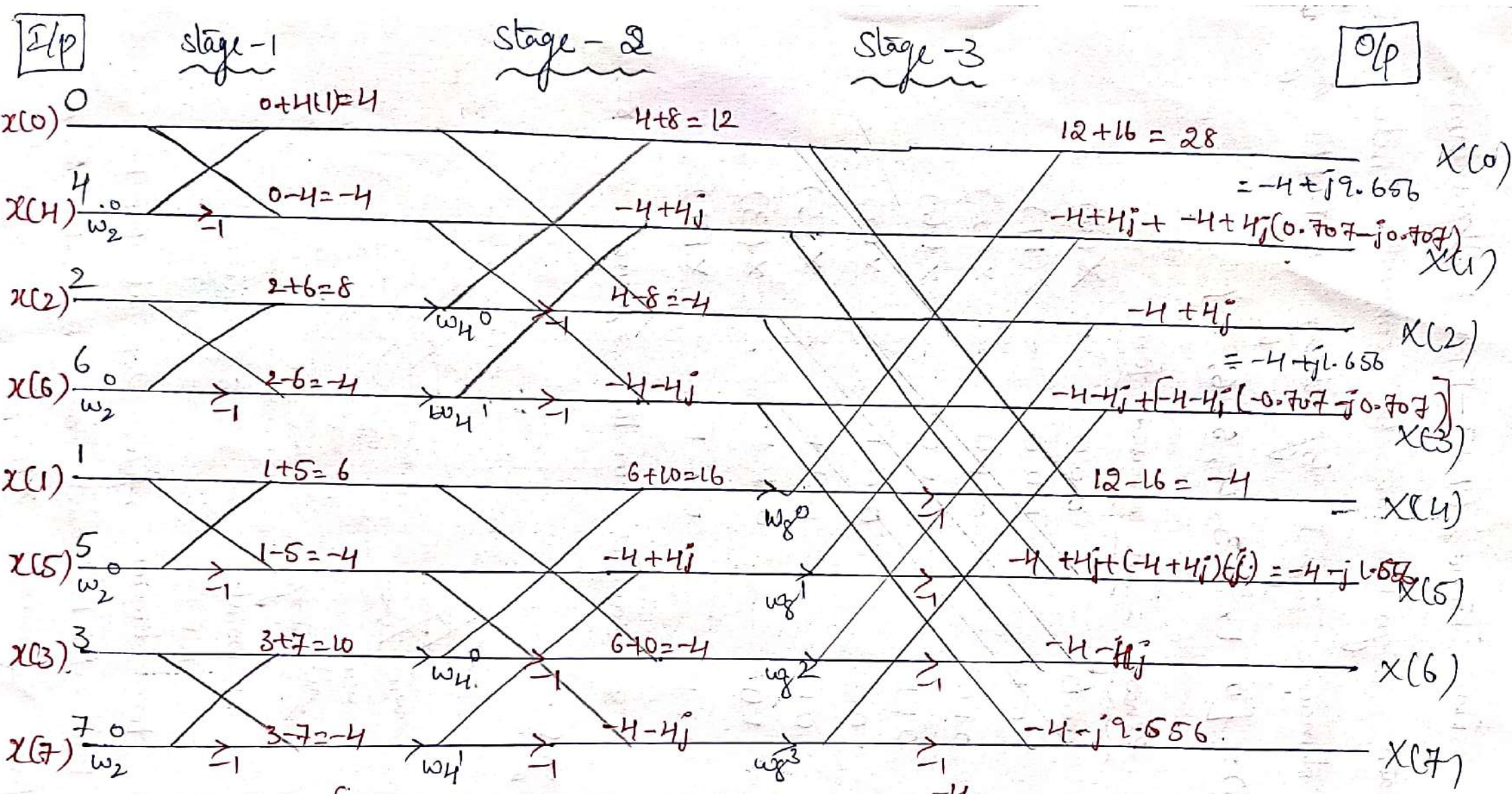


1-stage — 2-Point DFT

2nd-stage — 4-point DFT

3rd-stage — 8-point DFT.

* We need to draw four 2-point DFT's, Two Four point DFT's and one 8-point DFT. for the given sequence $x(n)$.



$\therefore X(K) = \{28, -4+j9.656, -4+j4, -4+j1.565, -4-4j1.656, -4-4j, -4-j9.565\}$

$-4,$

$\omega_2^0 / \omega_4^0 / \omega_8^0 = 1 \quad ; \quad \omega_8^1 = 0.707 - j0.707$

$\omega_4^2 = \omega_8^2 = -j \quad ; \quad \omega^3 = -0.707 - j0.707.$

Decimation in Frequency radix-2 (DIF-FFT) Algorithm.

Eg:- Find the 8-point DFT of a sequence $x(n) = \{1,1,1,1,0,0,0,0\}$ using radix 2 DIF-FFT Algorithm.

$\therefore X(K) = \{4, -j2.414, 0, 1-j0.414, 0, 1+j0.414, 0, 1+j2.414\}$

* The inputs for the sequence is $x(0)=1$, $x(1)=1$, $x(2)=1$, $x(3)=1$, $x(4)=x(5)=x(6)=x(7)=0$.

* Twiddle factors for this is $\omega_8^0 = 1$, $\omega_8^1 = 0.707 - j0.707$, $\omega_8^2 = -j$, $\omega_8^3 = -0.707 - j0.707$

(1) $0.707 - j0.707 + (-0.707 - j0.707)$

$= 0.707 - j0.707 - 0.707 - j0.707 = -j1.414$

(2) $0.707 - j0.707 + (-0.707 - j0.707)(-1)$

$= 0.707 - j0.707 + 0.707 + j0.707 = 1.414$

Differences between DIT-FFT and DIF-FFT Algorithm.

DIT-FFT	DIF-FFT
① The input is taken as bit reverse order and output is normal order.	① The input is taken as normal order and the o/p is bit reverse order.
② 3 stages of DFT (butterfies) are 2-point, 4-point & 8-point DFT will be calculated for 3 stages. DIT-FFT multiplication is calculated	② 3 stages of DFT (butterflies) are 8-point, 4-point & 2-point DFT will be calculated for 3 stages of DIF-FFT.
③ Twiddle factor before the each stage of DIT-FFT.	③ Twiddle factor multiplication is calculated after the each stage of DIF-FFT.

① Find the DFT of a sequence $x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$ using DIT-FFT. Algorithm.

Sol:- * place the input is bit reverse order i.e.,

$x(0), x(4), x(2), x(6), x(1), x(5), x(3), x(7)$.

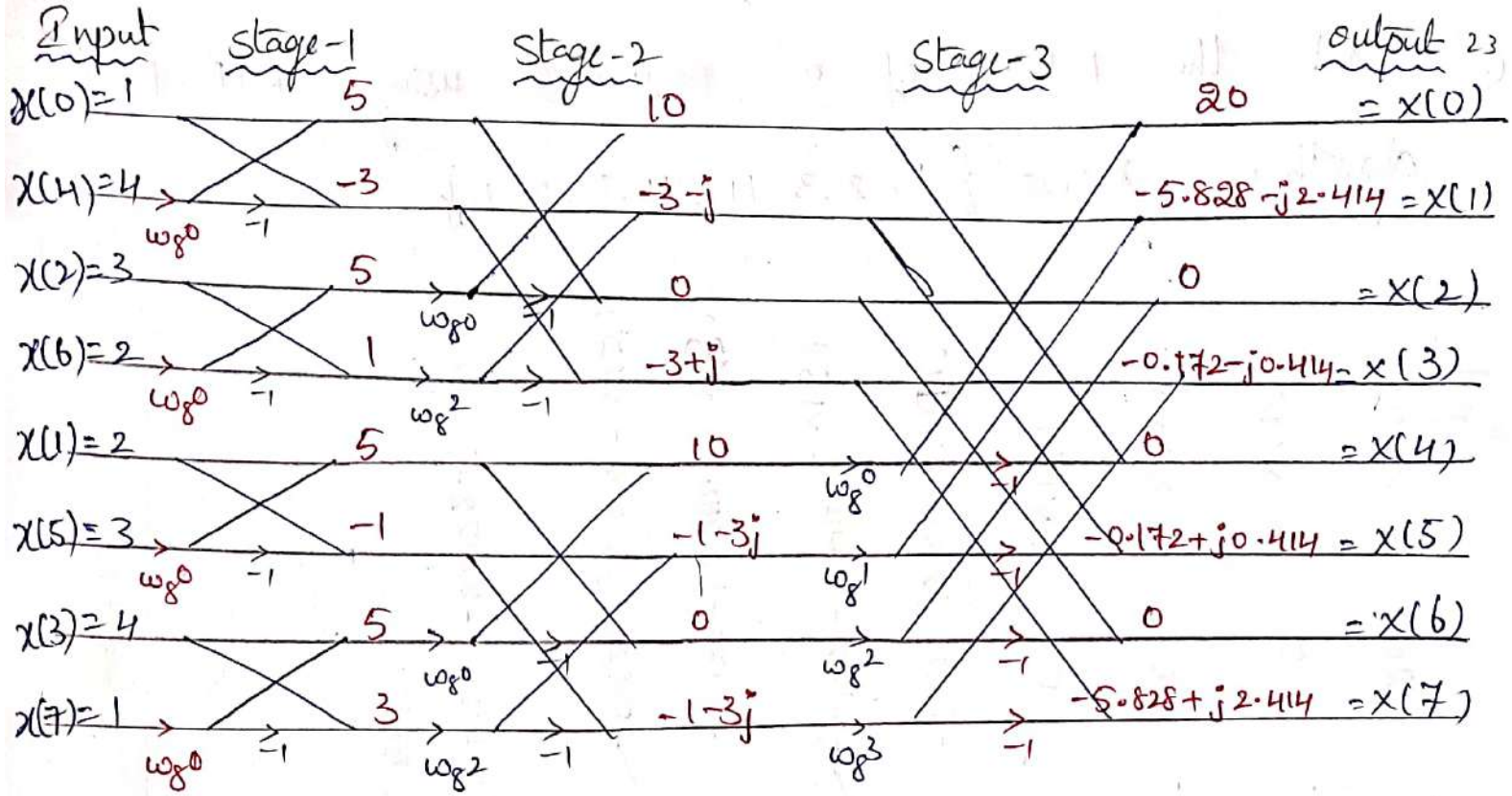
* Twiddle factors associated with the flow graph are

$$W_8^0 = 1$$

$$W_8^1 = 0.707 - j0.707$$

$$W_8^2 = -j$$

$$W_8^3 = -0.707 - j0.707.$$



$\therefore X(K) = \{20, -5.828-j2.414, 0, -0.172-j0.414, 0, -0.172+j0.414, 0, -5.828+j2.414\}$

Input	output of stage-1	output of stage-2	output
1 x(0)	1+4=5	5+5=10	10+10=20
4 x(4)	1-4=-3	-3+(-j)1 = -3-j	-3-j+(0.707-j0.707)(1-3j) = -5.828-j2.414
3 x(2)	3+2=5	5-5=0	0
2 x(6)	3-2=1	-3-(-j) = -3+j	(-3+j)+(0.707-j0.707)(-1+3j) = -0.172-j0.414
2 x(1)	2+3=5	5+5=10	10-10=0
3 x(5)	2-3=-1	-1+(j)3 = -1-3j	-3-j+(0.707-j0.707)(-1-3j) = -0.172+j0.414
4 x(3)	4+1=5	5-5=0	0
1 x(7)	4-1=3	-1-(-j)3 = -1+3j	(-3+j)-(-0.707-j0.707)(-1+3j) = -5.828+j2.414

(2) Find the DFT of a sequence using DIF-FFT Algorithm $x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$

$\therefore X(K) = \{20, -5.828 - j2.414, 0, -0.172 - j0.414, -5.828 - j2.414, -0.172 + j0.414, 0, -5.828 + j2.414\}$

=

Input	S_1	S_2	S_3 (o/p)	
$x(0)=1$	$1+4=5$	$5+5=10$	$10+10=20$	$X(0)$
$x(1)=2$	$2+3=5$	$5+5=10$	$10-10=0$	$X(4)$
$x(2)=3$	$3+2=5$	$5-5=0$	0	$X(2)$
$x(3)=4$	$4+1=5$	$(5-5)\omega_8^2=0$	0	$X(6)$
$x(4)=4$	$(1-4)\omega_8^0=-3$	$[-3+(-j)]=-3-j$	$-3-j-2.828-j1.414=-5.828-j2.414$	$X(1)$
$x(5)=3$	$(2-3)(0.707-j0.707)$ $=-0.707+j0.707$	$-0.707+j0.707+$ $(-2.121-j2.121)$ $=-2.828-j1.414$	$-3-j+2.828+j1.414=-0.172+j0.414$	$X(5)$
$x(6)=2$	$(3-2)(-j)=-j$	$-3-(-j)=-3+j$	$-3+j+2.828-j1.414$ $=-0.172-j0.414$	$X(3)$
$x(7)=1$	$(4-1)(-0.707-j0.707)$ $=-2.121-j2.121$	$(-0.707+j0.707+2.121$ $+j2.121(-j)$ $=2.828-j1.414$	$-3+j-2.828+j1.414$ $=-5.828+j2.414$	$X(7)$

$$x(k)=\{20,\ -5.828-j2.414,\ 0,\ -0.172-j0.414,\ 0,\ -0.172+j0.414,\ 0,\ -5.828+j2.414\}$$

(3) Compute the eight-point DFT of the sequence.

$$x(n) = \begin{cases} 1 & 0 \le n \le 7 \\ 0 & \text{otherwise} \end{cases}$$

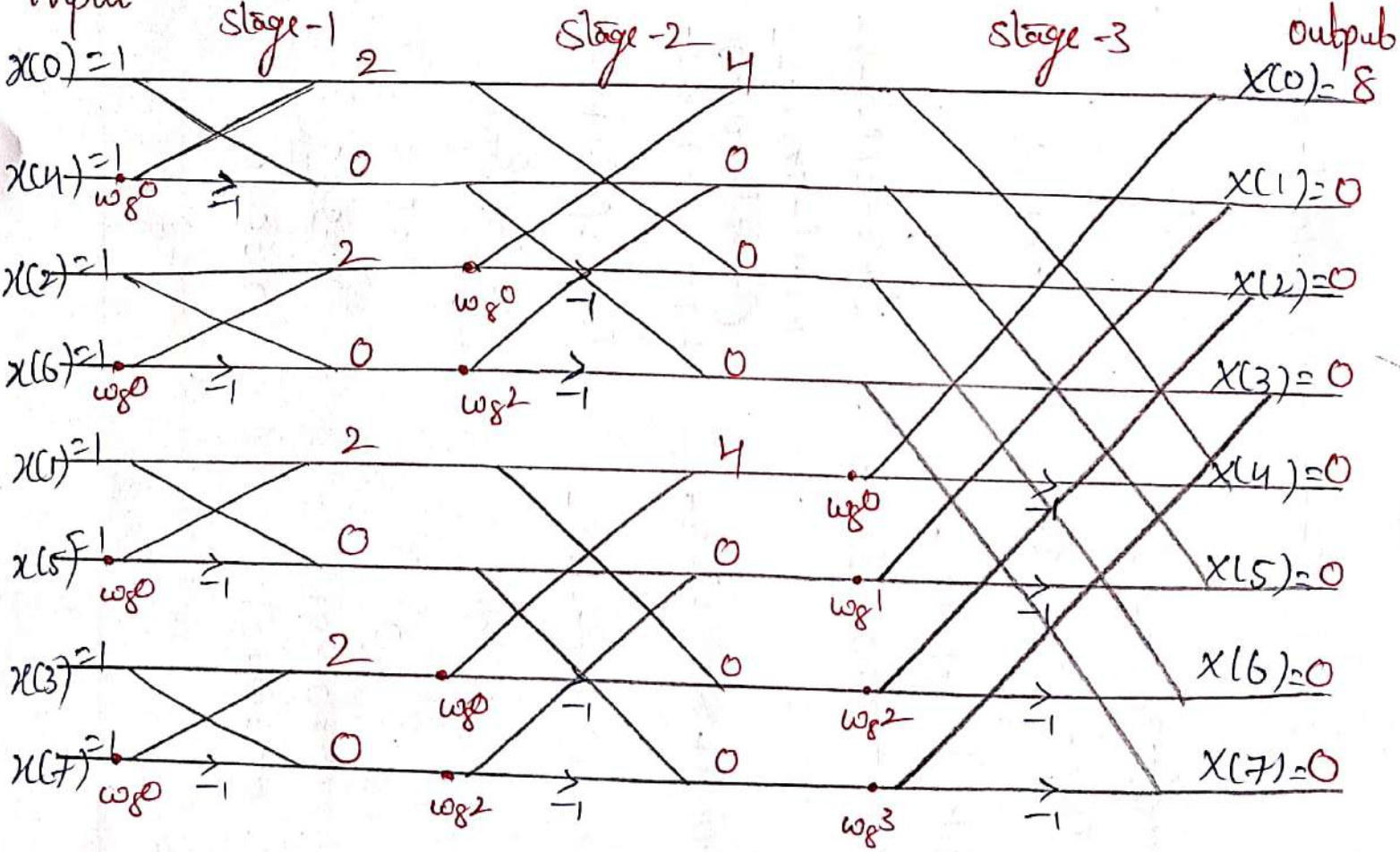
by using DIT, DIF Algorithm.

Sol:- DIT-FFT

* i/p's. $x(0)=1, x(1)=1, x(2)=1, x(3)=1, x(4)=1$
$x(5)=1, x(6)=1, x(7)=1$. (reverse bit order of i/p's)

* Twiddle factors associated with butterflies can be found as.

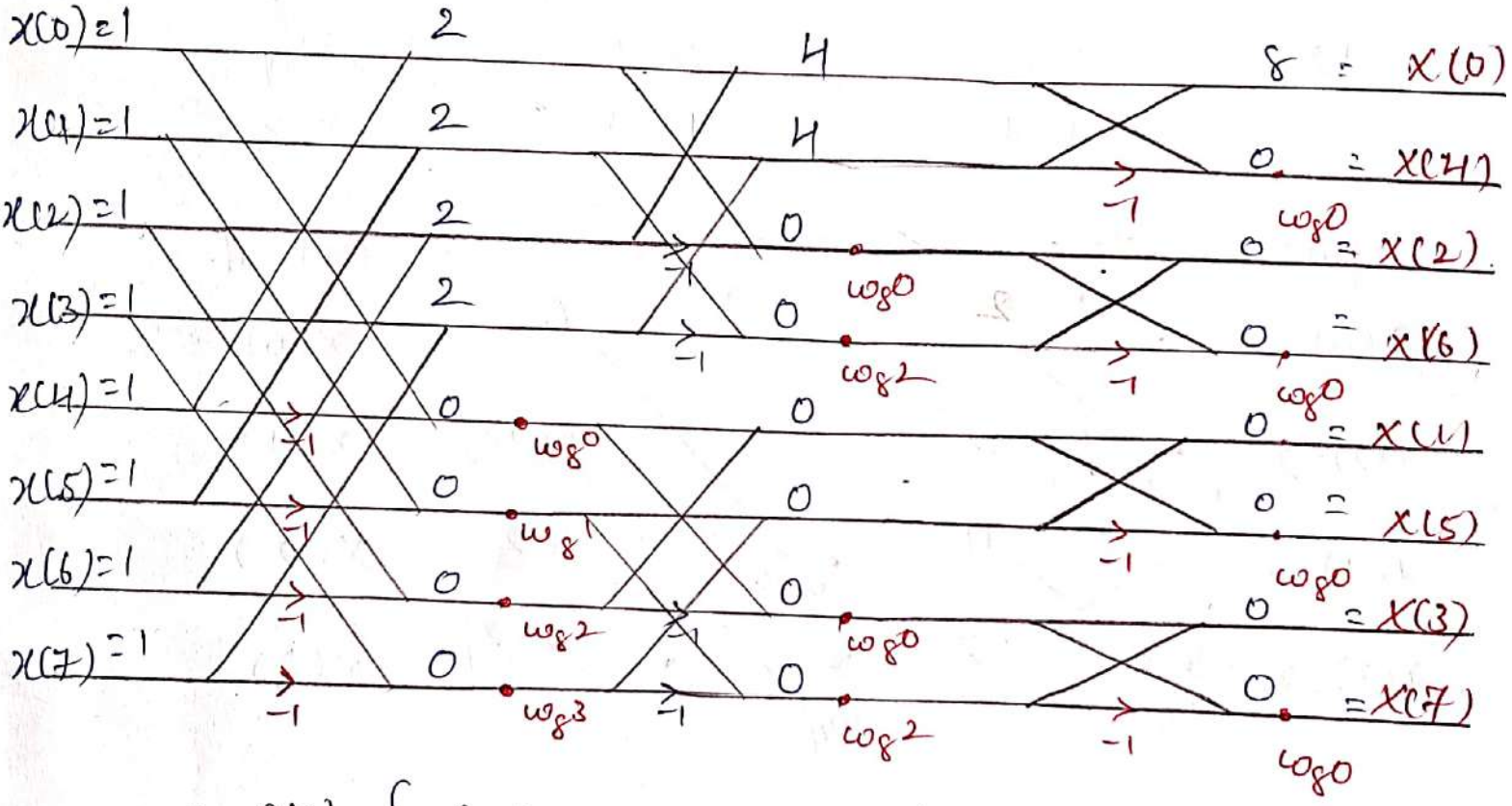
$W_8^0, W_8^1 = W_8^2, W_8^3$.



$\therefore X(K) = \{8,0,0,0,0,0,0,0\}$

DIF-FFT : $x(n) = \{1, 1, 1, 1, 1, 1, 1, 1\}$ 25

* Inputs $x(0)$ ---- $x(7) = 1$
* Twiddle factors $\omega_8^0, \omega_8^1, \omega_8^2, \omega_8^3$.



$\therefore x(k) = \{8, 0, 0, 0, 0, 0, 0, 0\}$

(4) compute 4-point DFT of a sequence $x(n) = \{0, 1, 2, 3\}$ using DIT-FFT and DIF-FFT Algorithm.

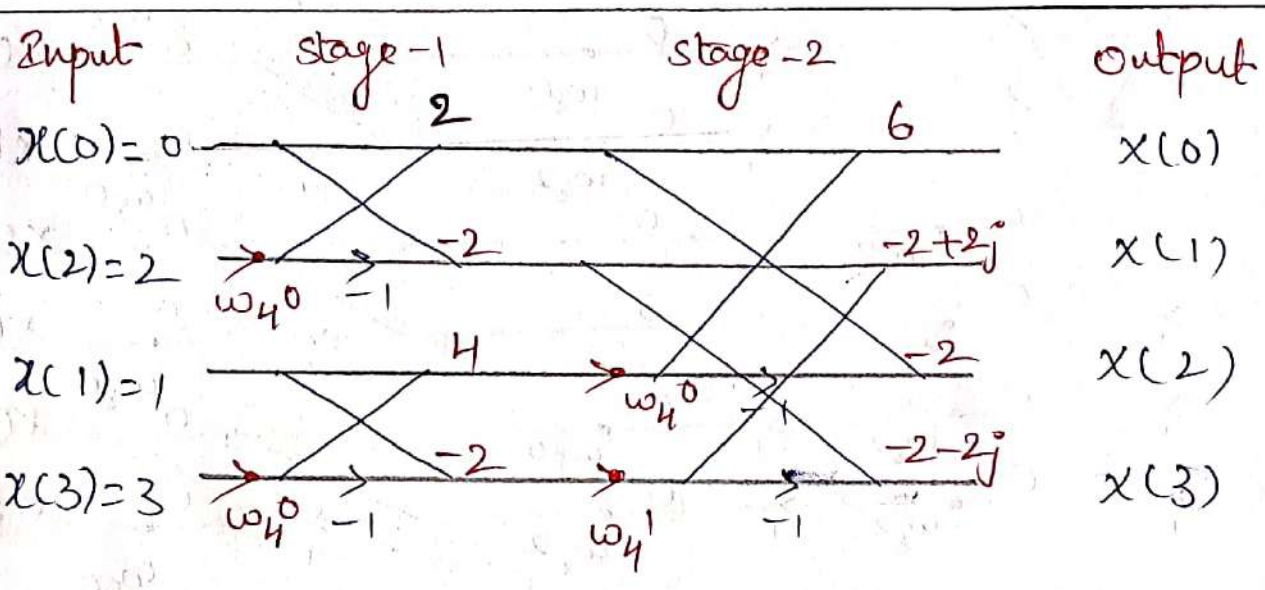
Sol:- DIT-FFT Algorithm:-

* Inputs for the sequence $x(0) = 1$, $x(1) = 1$, $x(2) = 2$ $x(3) = 3$.
* Twiddle factors associated with butterflies are

$\omega_4^0 = 1$, $\omega_4^1 = e^{-j2\pi/4} = e^{-j\pi/2} = -j$

Bit reversal of input is given by

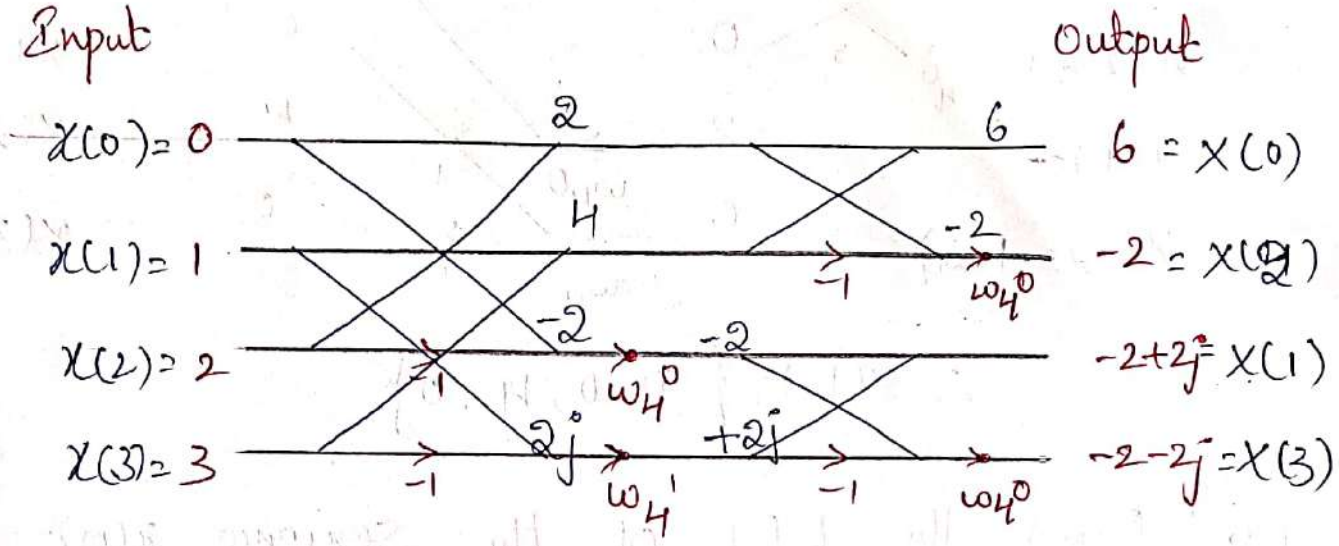
I/p index	Binary index	Bit-reversal	Bit-reversal index
0	0 0	0 0	0
1	0 1	1 0	2
2	1 0	0 1	1
3	1 1	1 1	3



Input	stage- S_1	stage-2
0	$0+2=2$	$2+4=6$
2	$0-2=-2$	$-2+(-j)(-2)=-2+2j$
1	$1+3=4$	$2-4=-2$
3	$1-3=-2$	$-2-(-j)(-2)=-2-2j$

$\therefore X(k) = \{6, -2+2j, -2, -2-2j\}$

DIF-FFT: $x(n) = \{0, 1, 2, 3\}$



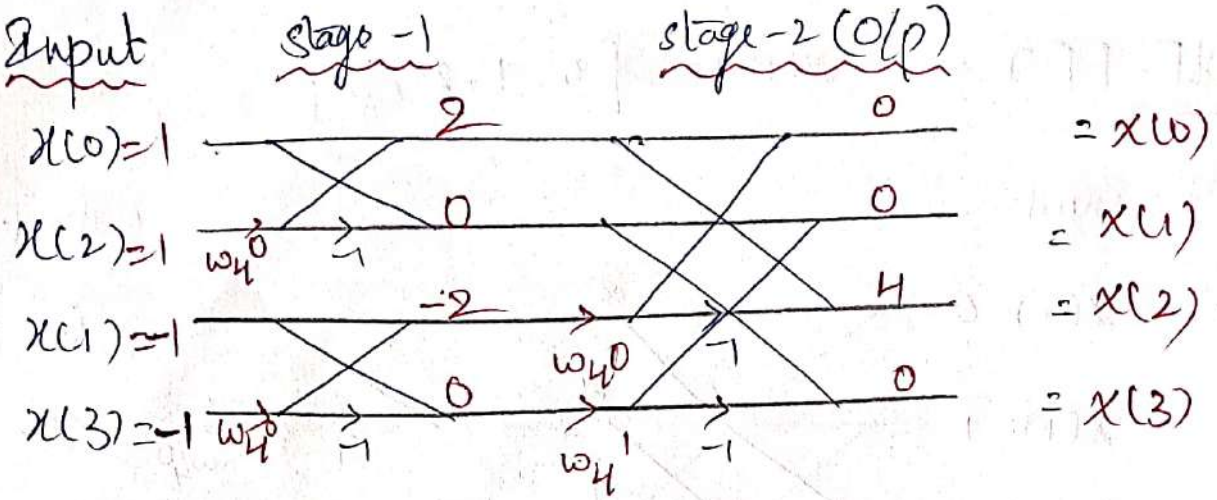
Input	S_1	Output
0	$0+2=2$	$2+4=6$
1	$1+3=4$	$2-4=-2$
2	$0-2=-2$	$-2+2j$
3	$(1-3)(-j)=2j$	$-2-2j$

$\therefore X(k) = \{6, -2+2j, -2, -2-2j\}$

(5) Compute the DFT of a sequence $x(n) = \{1, -1, 1, -1\}$ using DIT Algorithm.

Sol:- $x(n) = \{1, -1, 1, -1\}$

* Input's $x(0)=1$, $x(1)=-1$, $x(2)=1$, $x(3)=-1$.
* Twiddle factors ω_4^0, ω_4^1.

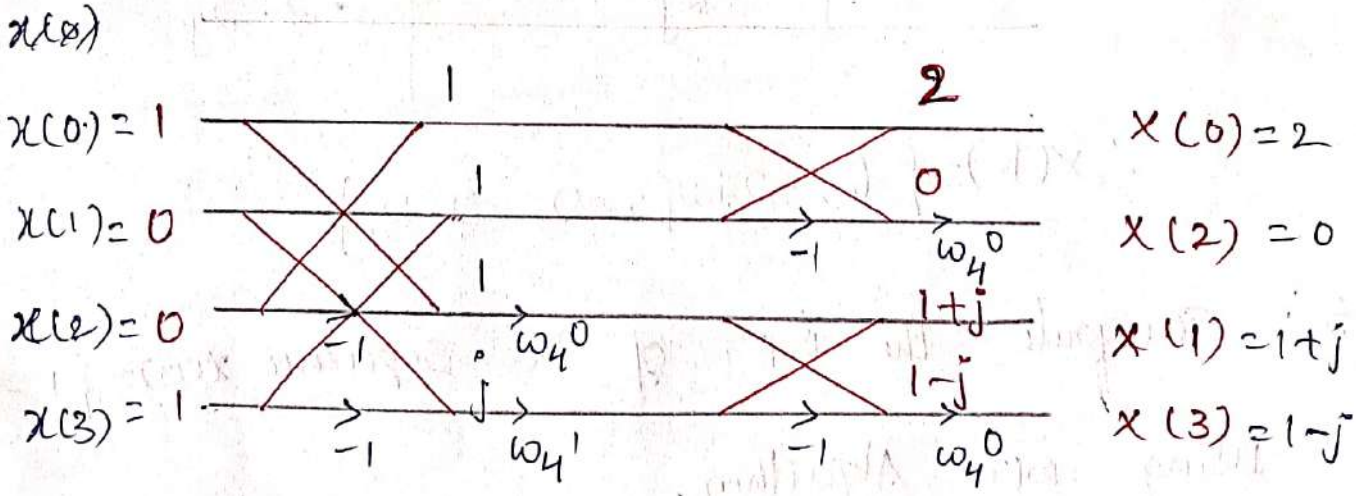


$\therefore x(K) = \{0, 0, 4, 0\}$

(6) Find the DFT of the sequence $x(n) = \{1, 0, 0, 1\}$ using DIF Algorithm.

Sol:- Inputs $x(0)=1$, $x(1)=0$, $x(2)=0$, $x(3)=1$

Twiddle factors associated with the sequence are $\omega_4^0 = 1$, $\omega_4^1 = -j$



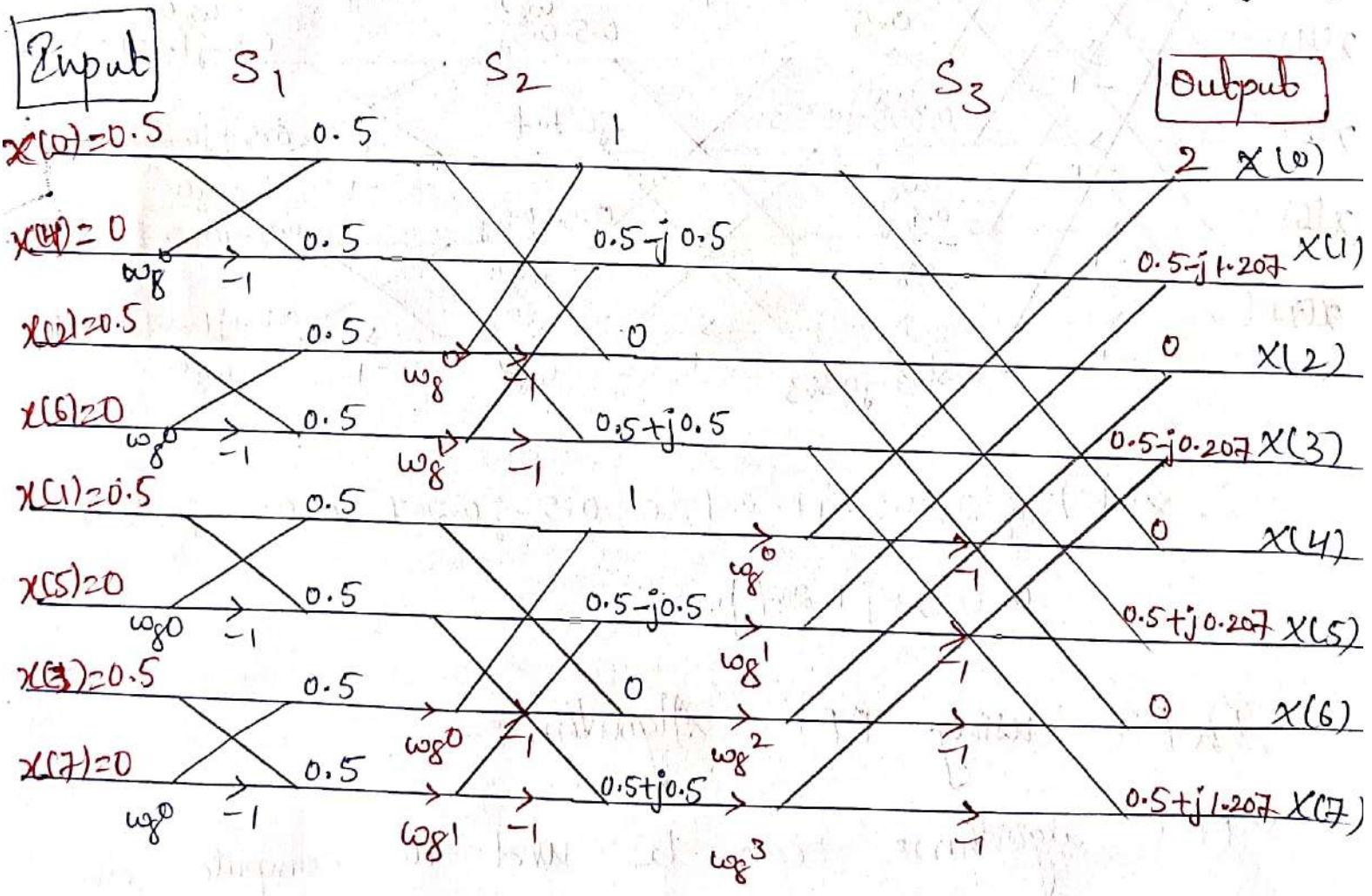
$\therefore x(K) = \{2, 1+j, 0, 1-j\}$

(7) Compute the eight-point DFT of the sequence $x(n) = \{0.5, 0.5, 0.5, 0.5, 0, 0, 0, 0\}$ using the inplace radix-2 DIT Algorithm.

Sol:- Inputs for the sequence is $x(0)=0.5$, $x(1)=0.5$, $x(2)=0.5$, $x(3)=0.5$, $x(4)=0$, $x(5)=0$, $x(6)=0$, $x(7)=0$.

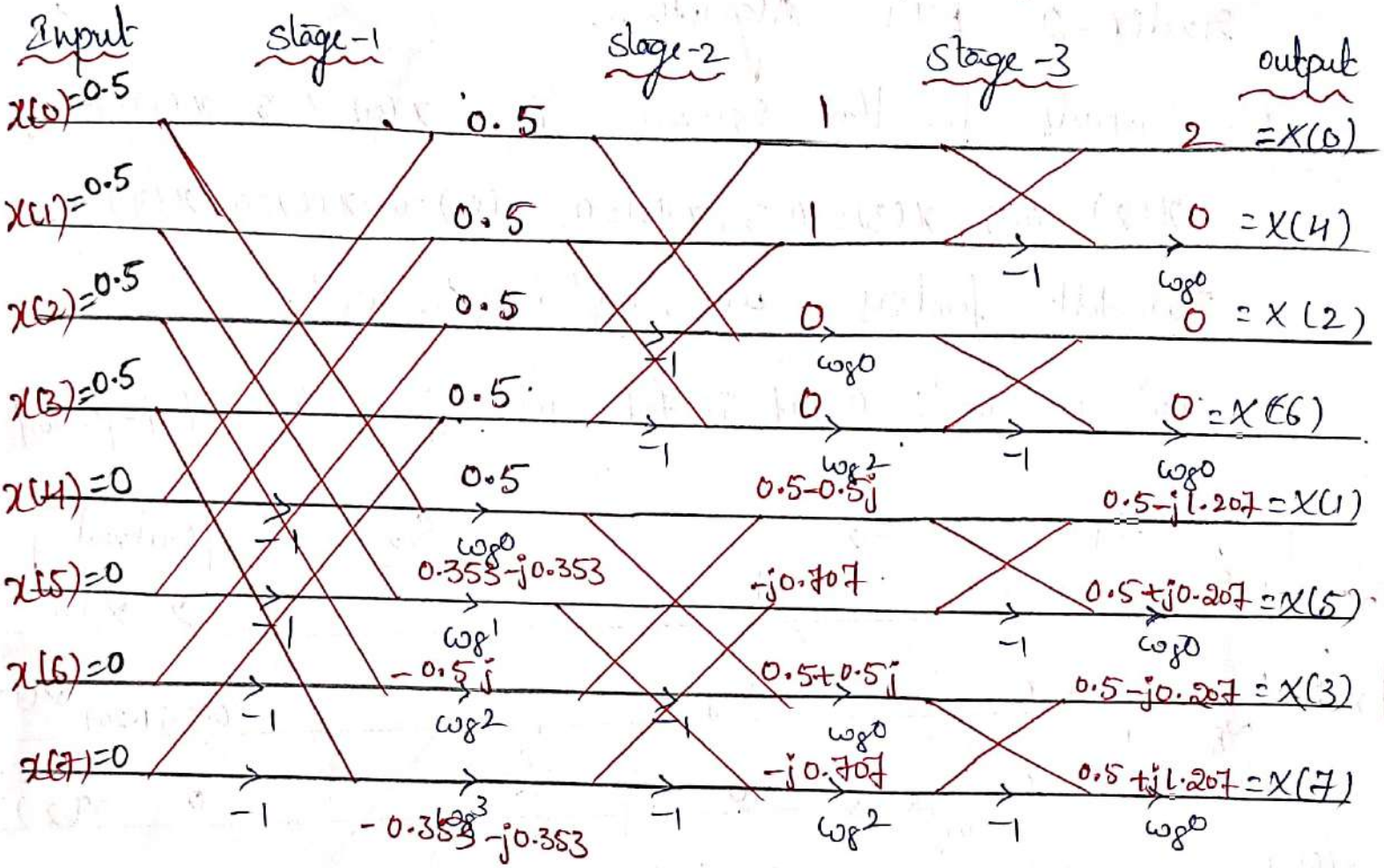
* Twiddle factors W_8^0, W_8^1, W_8^2, W_8^3.

$W_8^0 = 1$, $W_8^1 = 0.707 - j0.707$, $W_8^2 = -j$, $W_8^3 = -0.707 - j0.707$



$\therefore X(k) = \{2, 0.5-j1.207, 0, 0.5-j0.207, 0, 0.5+j0.207, 0, 0.5+j2.07\}$

(8) $x(n) = \{0.5, 0.5, 0.5, 0.5, 0, 0, 0, 0\}$. find the DFT of a given sequence using DIF Algorithm.



$\therefore X(K) = \{2, 0.5-j1.207, 0, 0.5-j0.207, 0, 0.5+j0.207, 0, 0.5+j1.207\}$

2. IDFT using FFT Algorithm:-

* FFT algorithms can be used to compute an inverse DFT without any change in the algorithm.

The inverse DFT of an N-point sequence $X(k)$, $k=0,1,\ldots N-1$ is defined as

$$x(n)=\frac{1}{N}\sum_{k=0}^{} X(k)W^{-nk} \longrightarrow (1)$$

where $W=e^{-j2\pi/N}$.

Take complex conjugate and multiply by 'N' we obtain

$$N x^*(n)=\sum_{k=0}^{N-1} X^*(k) W^{nk}. \longrightarrow (2)$$

The RHS of eqn (2) is DFT of sequence $X^*(k)$ and may be computed using any FFT algorithm. The desired output sequence $x(n)$ can then be found by complex conjugating the DFT and dividing by N to give

$$x(n)=\frac{1}{N}\left[\sum_{k=0}^{N-1} X^*(k) W^{nk}\right]^* \longrightarrow (3)$$

Steps for IFFT: [Finding IDFT of $X(k)$]

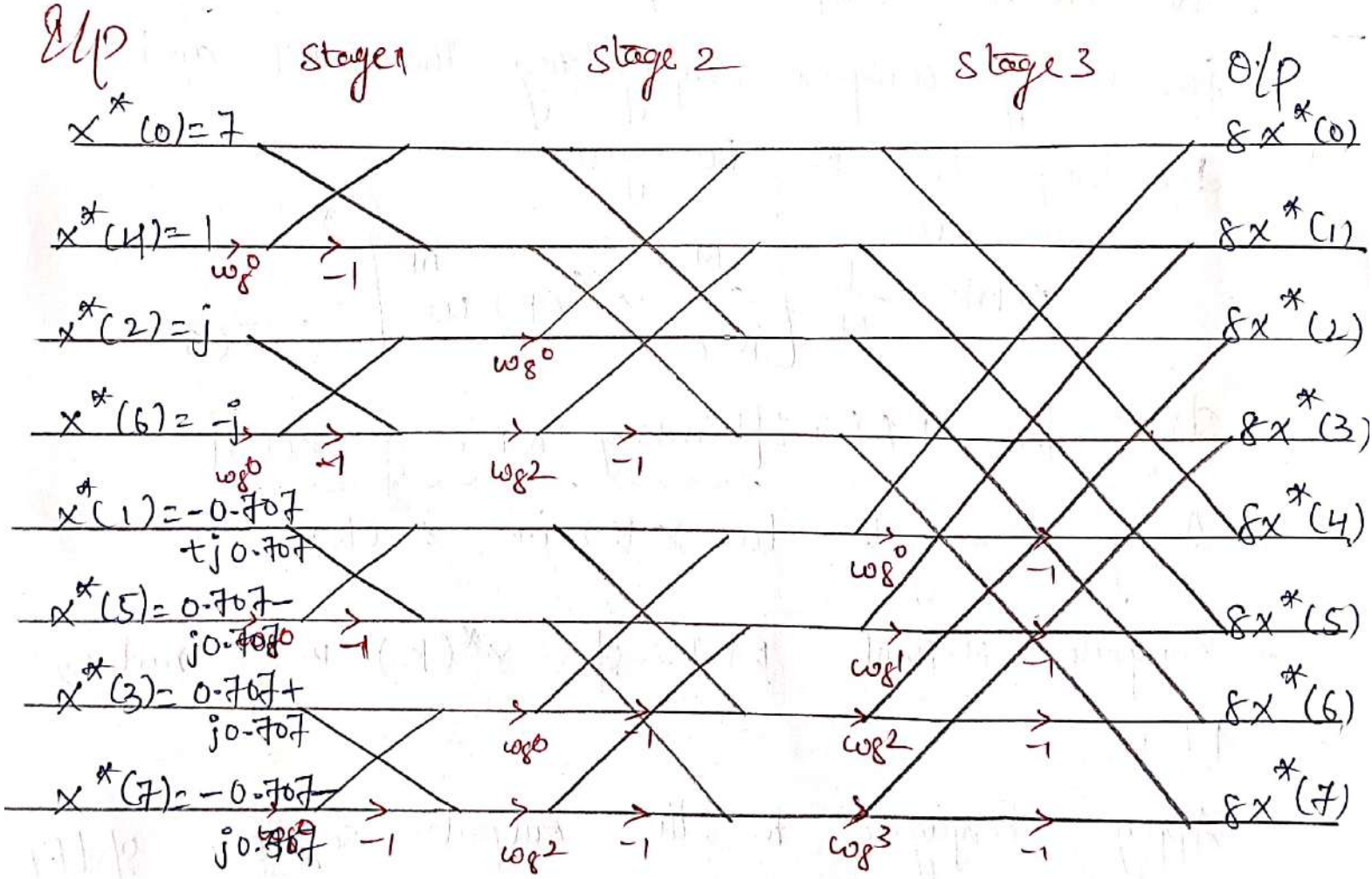
1. Apply conjugate to $X(k)$ i.e, $X^*(k)$
2. Compute N-point DFT of $X^*(k)$ using radix-2 FFT.
3. Apply conjugate to the output sequence of FFT.
4. Divide the output sequence of FFT by N.

① Compute 8-DFT of the sequence

$x(k) = \{7, -0.707 - j0.707, -j, 0.707 - j0.707, 1, 0.707 + j0.707, j, -0.707 + j0.707\}$.

Using DIT Algorithm.

Sol:- Take complex conjugate of $x(k)$ and apply bit reversal index inputs to flow graph of 8-point DIT-Algorithm.

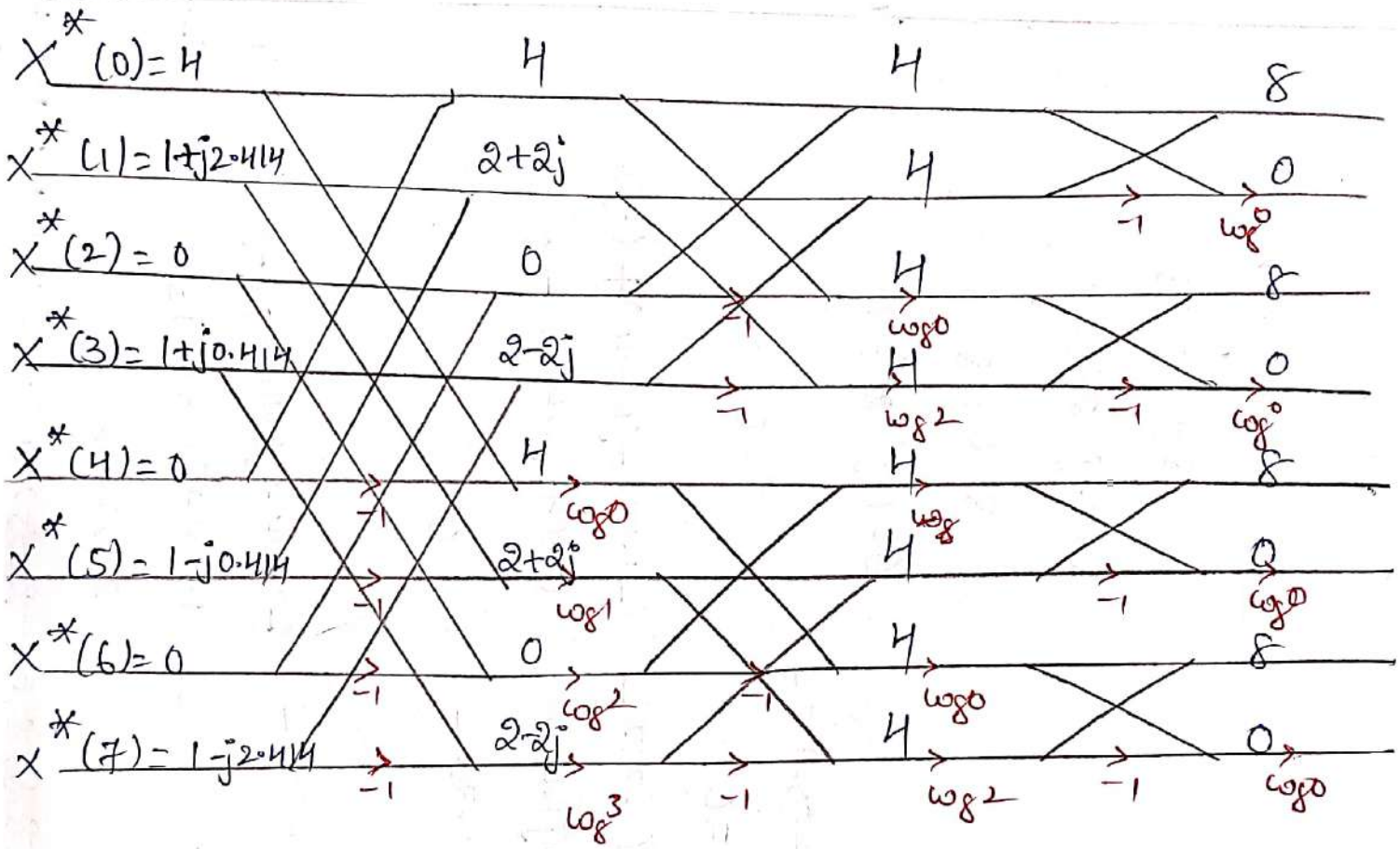


Input	stage 1	Stage 2	output
7	$7+1=8$	$8+0=8$	$8+0=8$
1	$7-1=6$	$6+(-j)2j=8$	$8+0(w^1)=8$
j	$j-j=0$	$8-0=8$	$8+0(w^2)=8$
$-j$	$j-(-j)=2j$	$6-(-j)2j=4$	$4+(0.707-j0.707)(-2.828+j2.828)=8$
$0.707+j0.707$	$(-0.707+j0.707)+(0.707-j0.707)=0$	0	$8-0=8$
$0.707-j0.707$	$(-0.707+j0.707)-(0.707-j0.707)=-1.414+j1.414$	$(-1.414+j1.414)+(-j)$ $(1.414+j1.414)=0$	$8-0(w^1)=8$
$0.707+j0.707$	$(0.707+j0.707)+(-0.707-j0.707)=0$	0	$8-0(w^2)=8$
$-0.707-j0.707$	$0.707+j0.707-(-0.707-j0.707)$ $=1.414+j1.414$	$(-1.414+j1.414)-(-j)$ $(1.414+j1.414)=$ $-2.828+j2.828$	$4-(-0.707-j0.707)\cdot(-2.828+j2.828)=0$
output	$x^*(n)=\{8,8,8,8,8,8,8,0\}$		$\Rightarrow x(n)=\{1,1,1,1,1,1,1,0\}$

(2) Find the 8DFT of the sequence $x(k) = \{4, 1-2j.414, 0, 1-j0.414, 0, 1+j0.414, 0, 1+j2.414\}$ using DIF Algorithm.

Sol:- Inputs — $x(0)$ ------ $x(7)$

Twiddle factors — W_8^0, W_8^1, W_8^2 & W_8^3.



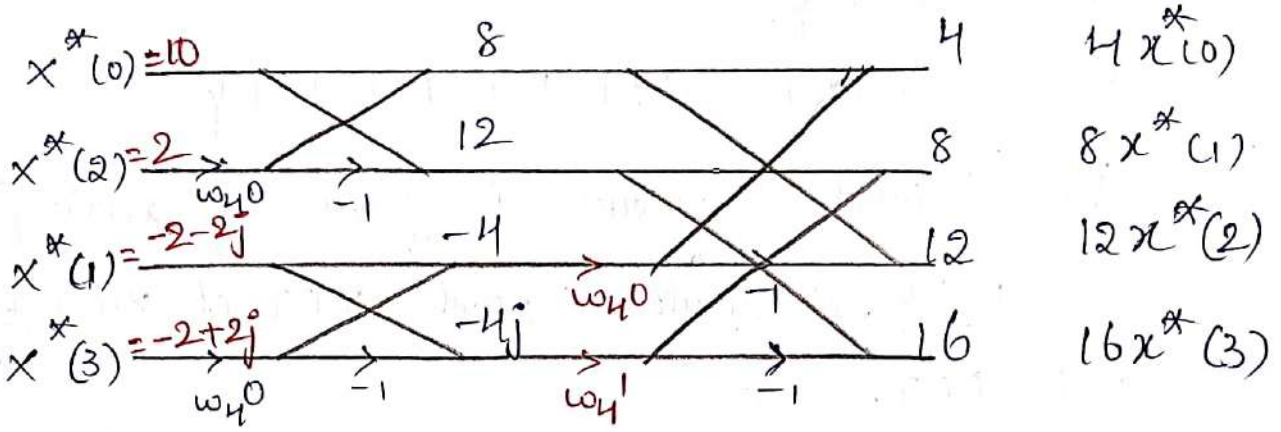
The output $x^*(n)$ is in bit reversal order.

Therefore

$$x(n) = \{1, 1, 1, 1, 0, 0, 0, 0\}$$

(3) Find the 8DFT of the sequence $x(k) = \{10, -2+2j, -2, -2-2j\}$ using DIT Algorithm.

Sol:- The twiddle factors are $W_4^0 = 1$, $W_4^1 = -j$

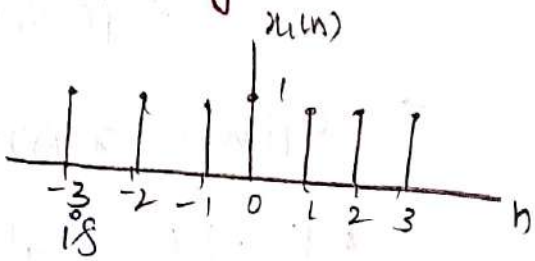


The output $N x^*(n)$ is normal order.

Therefore $x(n) = \{1, 2, 3, 4\}$

* Evaluate and compare the 8-point DFT for the following sequences using DIT-FFT algorithm.

(a) $x_1(n) = \begin{cases} 1 & \text{for } -3 \le n \le 3 \\ 0 & \text{otherwise.} \end{cases}$

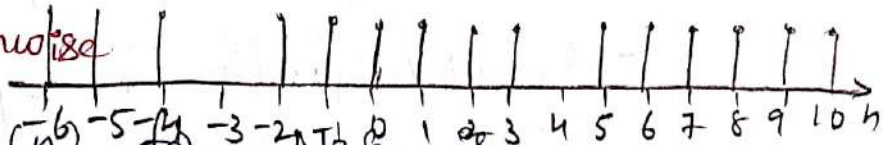


The sequence $x_1(n)$ for $N=8$ is

$x_1(n) = \{1, 1, 1, 1, 0, 1, 1, 1\}$

Ans:- $X_1(K) = \{7, 1, -1, 1, -1, 1, -1, 1\}$

(b) $x_2(n) = \begin{cases} 1 & \text{for } 0 \le n \le 6 \\ 0 & \text{otherwise} \end{cases}$



The sequence $x_2(n)$ for $N=8$.

$x_2(n) = \{1, 1, 1, 1, 1, 1, 1, 0\}$

$x_2(n)$ can be obtained by shifting $x_1(n)$ right 3 times. Using Time shifting property

Ans:- $X_2(K) = \{7, -0.707 - j0.707, -j, -0.707 - j0.707, 1, 0.707 + j0.707, j, -0.707 + j0.707\}$

* Compute the FFT for the sequence $x(n) = \{1, 0, 0, 0, 0, 0, 0, 0\}$.

Ans:- $X(K) = \{1, 1, 1, 1, 1, 1, 1, 1\}$

* An 8-point sequence is given by $x(n) = \{2, 2, 2, 2, 1, 1, 1, 1\}$, Compute 8-point DFT of $x(n)$ by radix-2 DIT FFT.

Ans:- $X(K) = \{12, 1-j2.414, 0, 1-j0.414, 0, 1+j0.414, 0, 1+j2.414\}$

* Find the DFT of the sequence $x(n) = \{1, -1, -1, -1, 1, 1, 1, -1\}$. using DIT-FFT Algorithm.

Ans:- $X(K) = \{0, -1.414+j3.414, 2-j2, 1.414-j0.586, 4, 1.414+j0.586, 2+2j, -1.414-j3.414\}$.

* Given $x(n) = 2^n$ and $N=8$, find $X(K)$ using DIT-FFT algorithm.

Sol:- Given $x(n) = 2^n$.

$\therefore x(n) = \{1, 2, 4, 8, 16, 32, 64, 128\}$

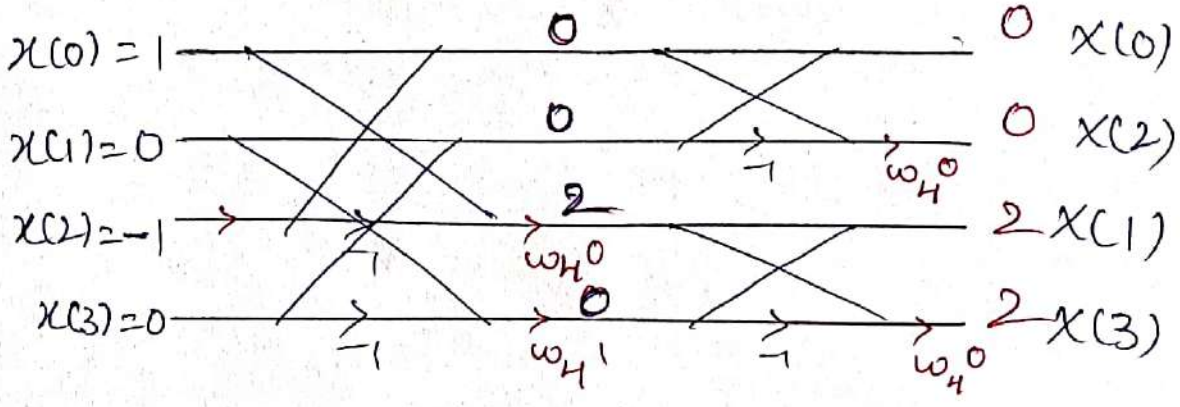
Ans:- $X(K) = \{255, 48.63+j166, -51+j102, -78.63+j46.05, -85, -78.63-j46.05, -51-j102, 48.63-j166\}$

* Compute the DFT of the sequence $x(n) = \cos\frac{n\pi}{2}$ where $N=4$. Using DIF-FFT Algorithm.

Sol:- Given $x(n) = \cos\frac{n\pi}{2}$ $x(0)=1, x(1)=0,$

$x(2)=-1, x(3)=0$

$\therefore x(n) = \{1, 0, -1, 0\}$



$\therefore x(K) = \{0, 2, 0, 2\}$

* Find the 8-point DFT of the sequence $x(n) = \{1, 2, 2, 1, 1, 2, 2, 1\}$ using DIF-FFT Algorithm

Ans:- $\{12, 0, 2, -j2, 0, 0, 0, -2+2j, 0\}$

* Find the IDFT of the sequence.

$x(K) = \{1, 1+j, 2, 1-2j, 0, 1+2j, 0, 1-j\}$.

Ans:- $\{0.875, 3.018 + j0.25, -0.870, 0.3018 - j0.25,$

$-0.125, -0.0518 + j0.25, 0.625, -0.0518 - j0.25\}$

* Find the 8-point DFT of the sequence $x(n) = \{0, 1, 2, 3, 4, 5, 6, 7\}$ using DIF, radix-2 FFT Algorithm.

Ans:- $x(K) = \{28, -4 + j9.656, -4 + j4, -4 + j1.656, -4,$

$-4 - j1.656, -4 - 4j, -4 - j9.656\}$.

— END —

UNIT - III

INFINITE IMPULSE RESPONSE FILTERS

Introduction to digital Filters :-

Digital filter is a linear time-invariant discrete time system. The term Finite Impulse Response (FIR) and Infinite Impulse Response (IIR) are used to distinguish a filter types.

FIR filters are of non-recursive type, where by the present output sample depends on the present input sample and previous input samples.

Where as the IIR filters are of recursive type, where by the present output sample depends on the present input, past input samples and output samples.

The Impulse response $h(n)$ for a realizable filter is

$$h(n) = 0 \quad \text{for} \quad n \leq 0 \longrightarrow (1)$$

and for stability it must satisfy the condition.

$$\sum_{n=0}^{\infty} |h(n)| < \infty \longrightarrow (2)$$

IIR digital filters have the transfer function of the form

$$H(Z) = \sum_{n=0}^{\infty} h(n) \cdot z^{-n} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k \cdot z^{-k}} \longrightarrow (3)$$

Frequency Selective Networks:-

Low Pass Filter:- The magnitude response of an ideal LPF allows LF in the pass band $0 < \Omega < \Omega_c$ to pass, where as the higher frequencies in the stopband $\Omega > \Omega_c$ are blocked. The frequency Ω_c btw the two bands is the cutoff frequency. where the magnitude response $|H(j\Omega)| = 1/\sqrt{2}$.

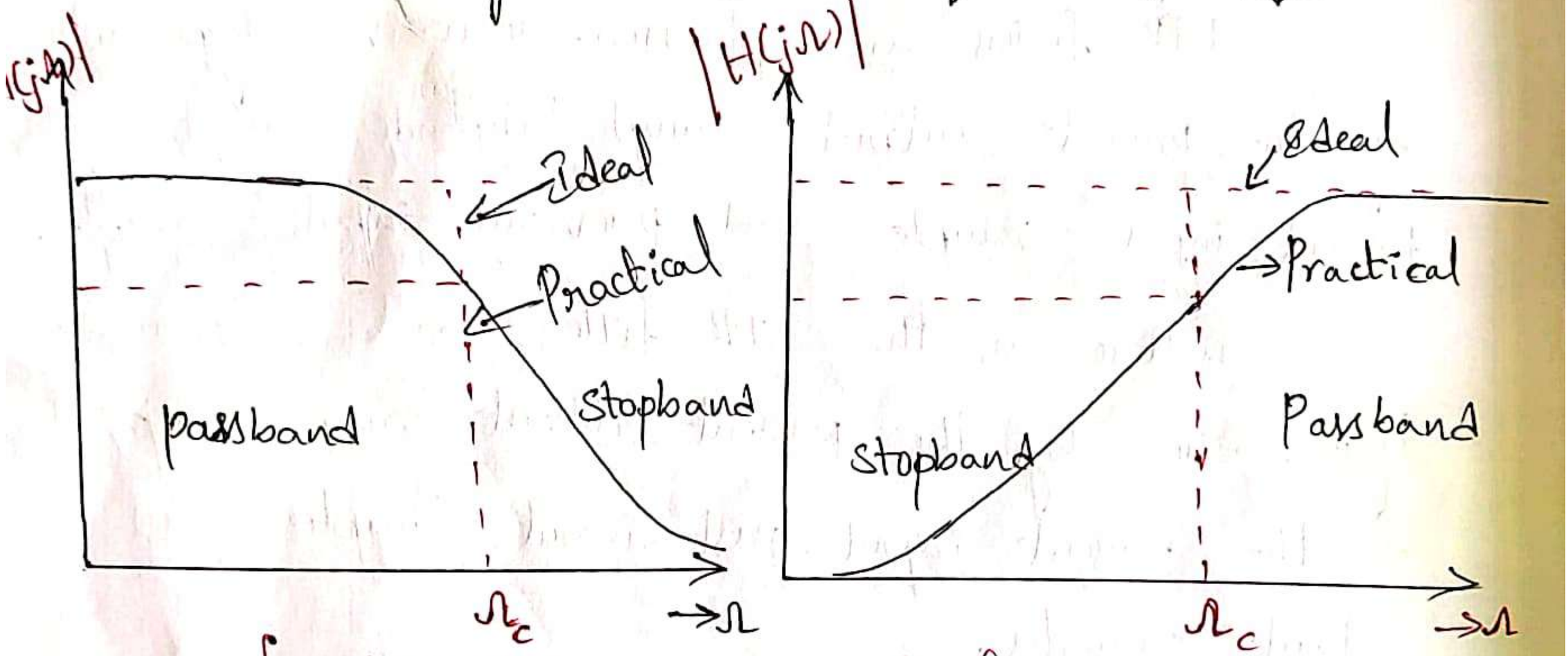
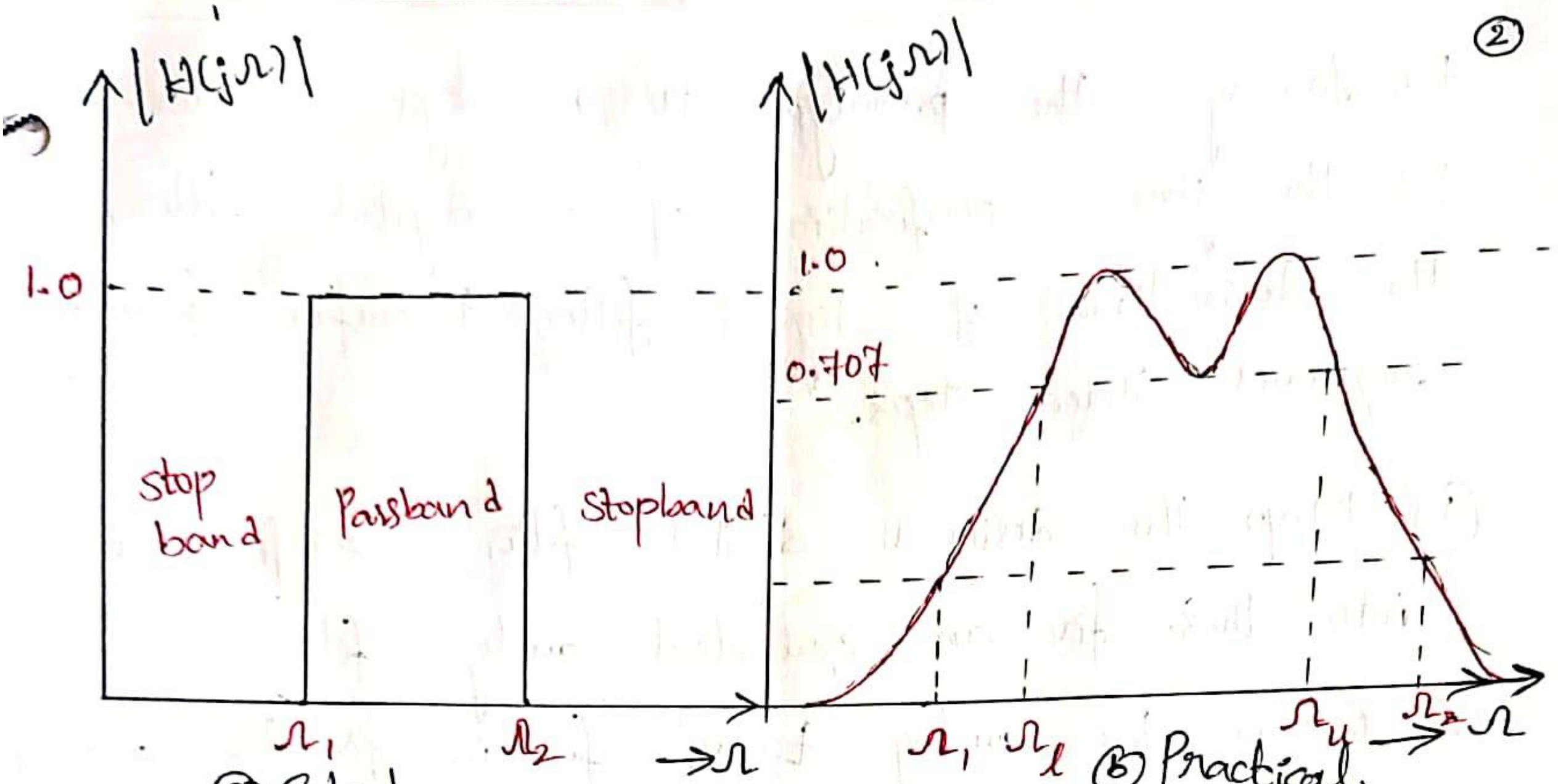


Fig: Magnitude response of filters

(a) LPF (b) HPF

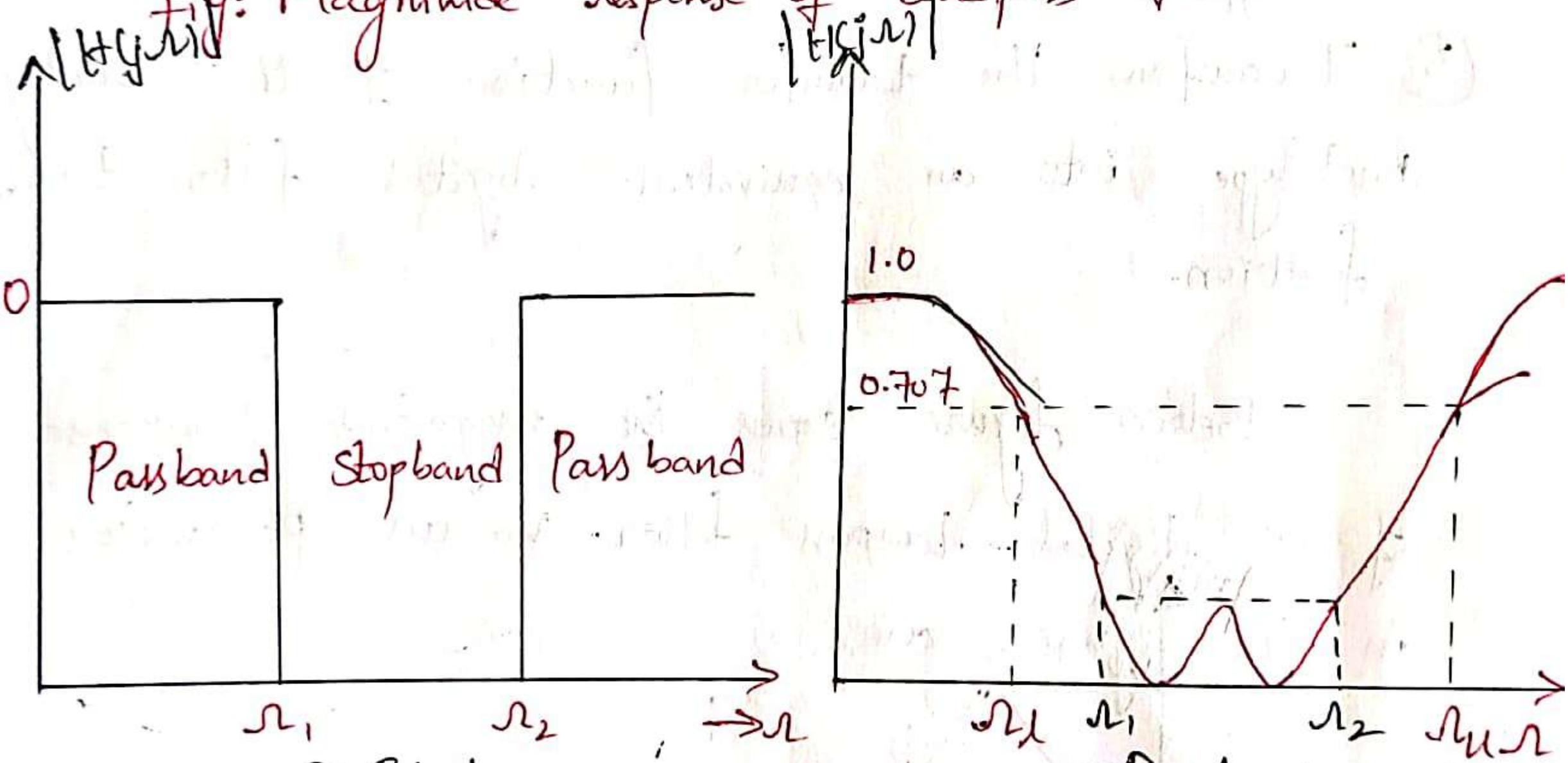
High pass Filter:- High pass filter allows the frequency above $\Omega > \Omega_c$ & reject frequency btw $\Omega = 0$ & $\Omega = \Omega_c$.

Bandpass filter:- It allows only a band of frequencies Ω_1 to Ω_2 to pass and stops all other frequencies.



ⓐ Ideal ⓑ Practical.

fig: Magnitude response of Bandpass filter



ⓐ Ideal ⓑ Practical

Fig:- Magnitude response of Bandreject filter

* Design of Digital filters from Analog filters:-

The most common technique used for designing IIR digital filters known as indirect method, involves first designing an analog prototype filter and then

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transforming the prototype to a digital filter.

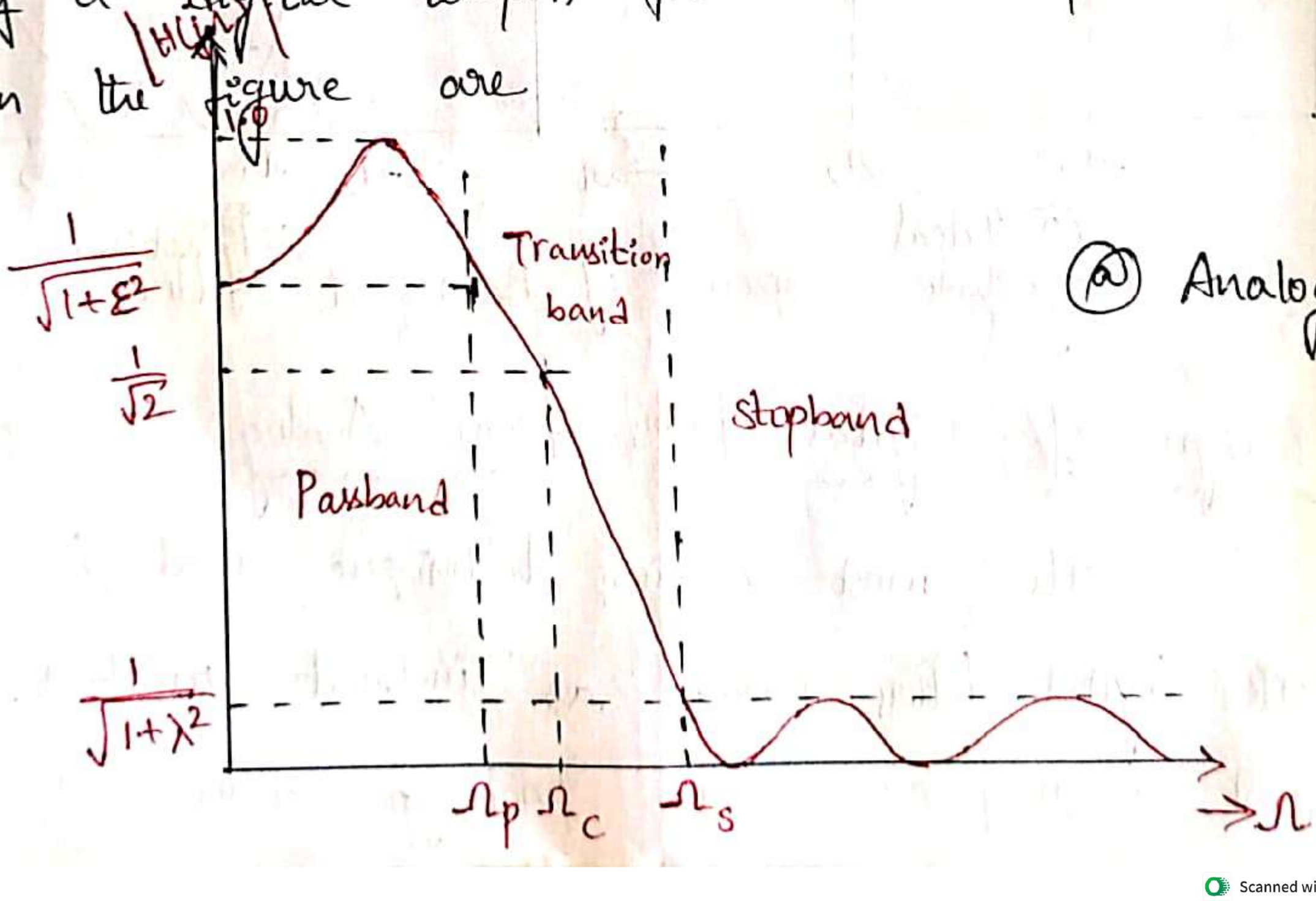
For the given specifications of a digital filter, the derivation of digital filter transfer function requires three steps.

(1) Map the desired digital filter specifications into those for an equivalent analog filter.

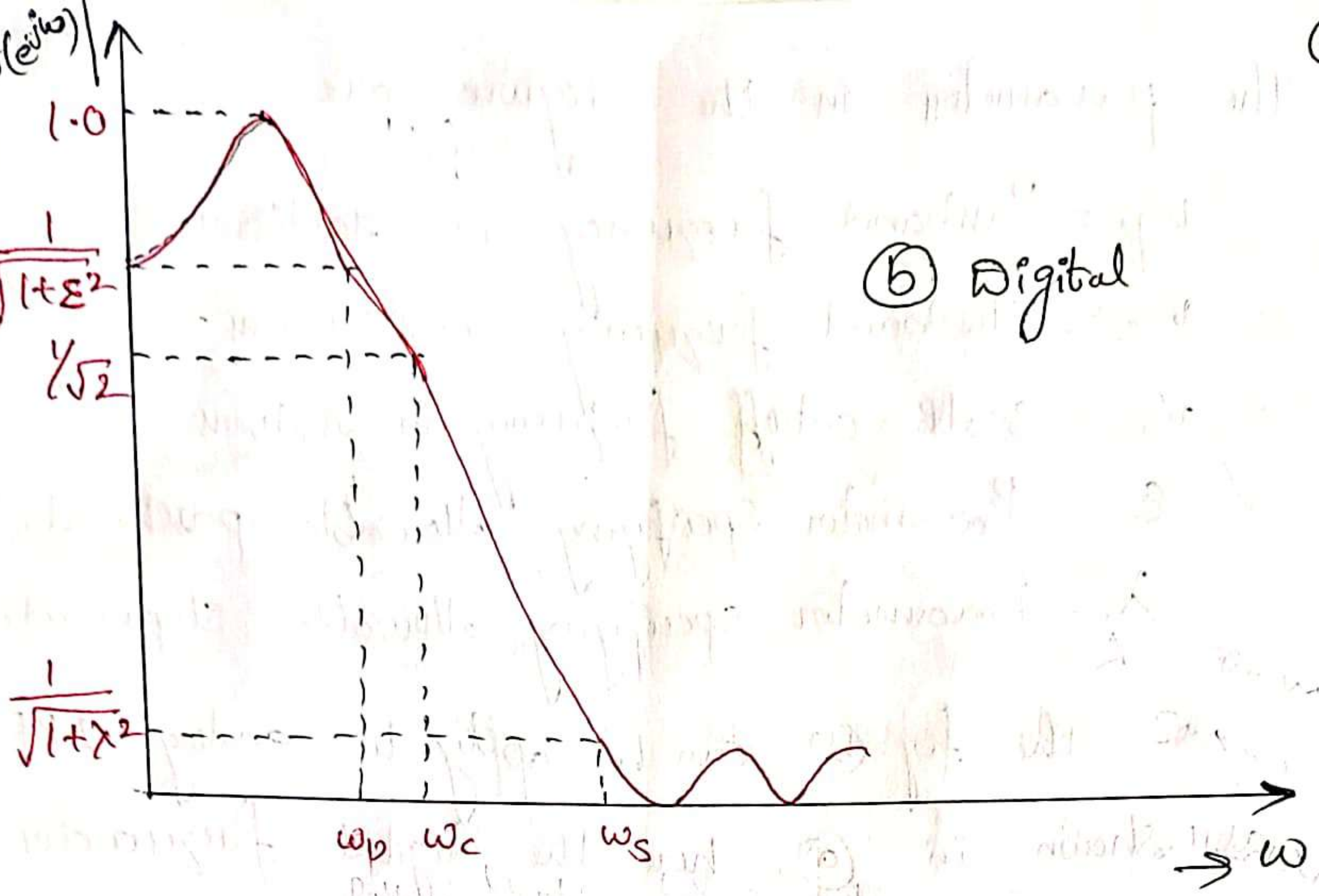
(2) Derive the analog transfer function for the analog prototype.

(3) Transform the transfer function of the analog prototype into an equivalent digital filter transfer function.

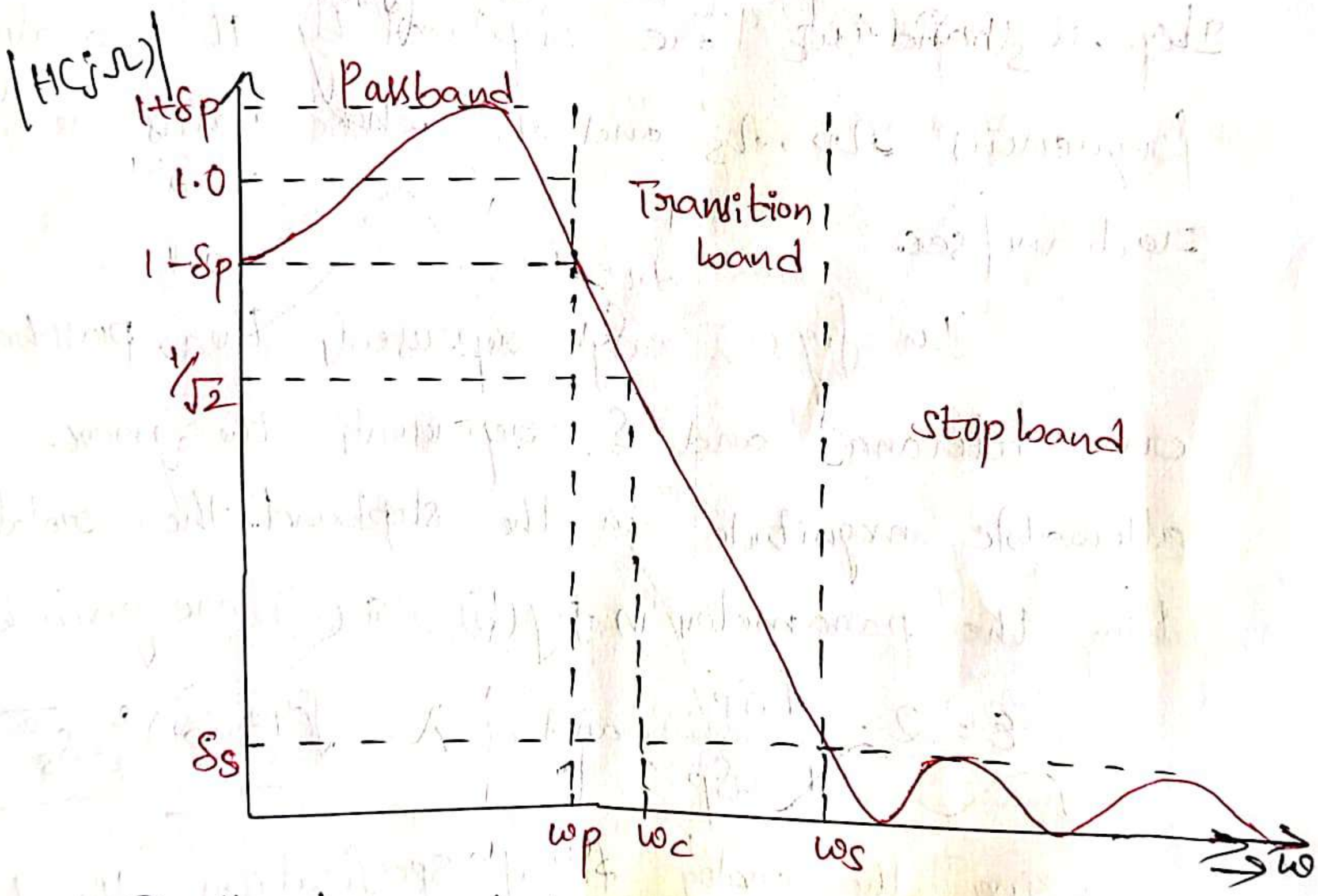
Below figure shows the magnitude response of a digital lowpass filter. Various parameters in the figure are



(a) Analog



(b) Digital



(c) Alternate specifications of magnitude response of LPF.

Fig: Specifications for the magnitude response of LPF.

The parameters in the figure are

w_p = Passband frequency in radians

w_s = Stopband frequency in radians

w_c = 3-dB cutoff frequency in radians

ε = Parameter specifying allowable passband

λ = Parameter specifying allowable stopband.

The fig. (b) can be apply to analog LPF as shown in (a) here the digital frequencies w_p, w_s and w_c are replaced by the analog frequencies Ω_p, Ω_s and Ω_c whose units are in radians/sec.

In fig (c) δ_p represents the passband ripple tolerance and δ_s represents the maximum allowable magnitude in the stopband. The relation btw the parameters in fig (b) & (c) are given by

$$\varepsilon = 2 \cdot \frac{\sqrt{\delta_p}}{1-\delta_p} \quad \text{and} \quad \lambda = \frac{\sqrt{(1+\delta_p)^2 - \delta_s^2}}{\delta_s}$$

Using the analog filter specifications δ_s the LPF is designed & is transformed to digital filter using different methods, in that bilinear transformation is usually the method of choice for the design of filters like LPF, HPF, BPF & BRF.

Digital Versus Analog Filters:-

Analog Filter	Digital Filter
Analog filter processes analog inputs and generates analog outputs.	1. A digital filter processes and generates digital data.
Analog filters are constructed from active / passive electronic components.	2. A digital filter consists of elements like adder, multiplier & delay unit.
3. Analog filter is described by a differential equation.	3. Digital filter is described by difference equation.
4. The frequency response of an analog filter can be modified by changing the components.	4. The frequency response can be changed by changing the filter coefficients.

Analog Low Pass Filter Design:-

The most general form of analog filter transfer function is $H(S) = \frac{N(S)}{D(S)} = \frac{\sum_{i=0}^{M} a_i S^i}{1 + \sum_{i=1}^{N} b_i S^i}$ ⟶ (1)

where $H(S)$ is the laplace transform of the impulse response $h(t)$, that is

$$H(S) = \int_{-\infty}^{\infty} h(t) \cdot e^{-st} \, dt \longrightarrow (2)$$

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and $N \geq M$ must be satisfied. For a stable analog filter, the poles of $H(s)$ lie in the left half of the s-plane. Now we study two types of analog filter design. They are

① Butterworth Filter ② Chebyshev Filter.

Analog Lowpass Butterworth Filter:-

The magnitude function of the Butterworth LPF is given by

$$|H(j\Omega)| = \frac{1}{\left[1 + (\Omega/\Omega_c)^{2N}\right]^{1/2}}, \quad N = 1, 2, 3, \ldots$$

fig:- LPF Butterworth Magnitude Response of frequency in $\rightarrow \Omega$

Analog Butterworth Low Pass Filter:-

Steps to Design:

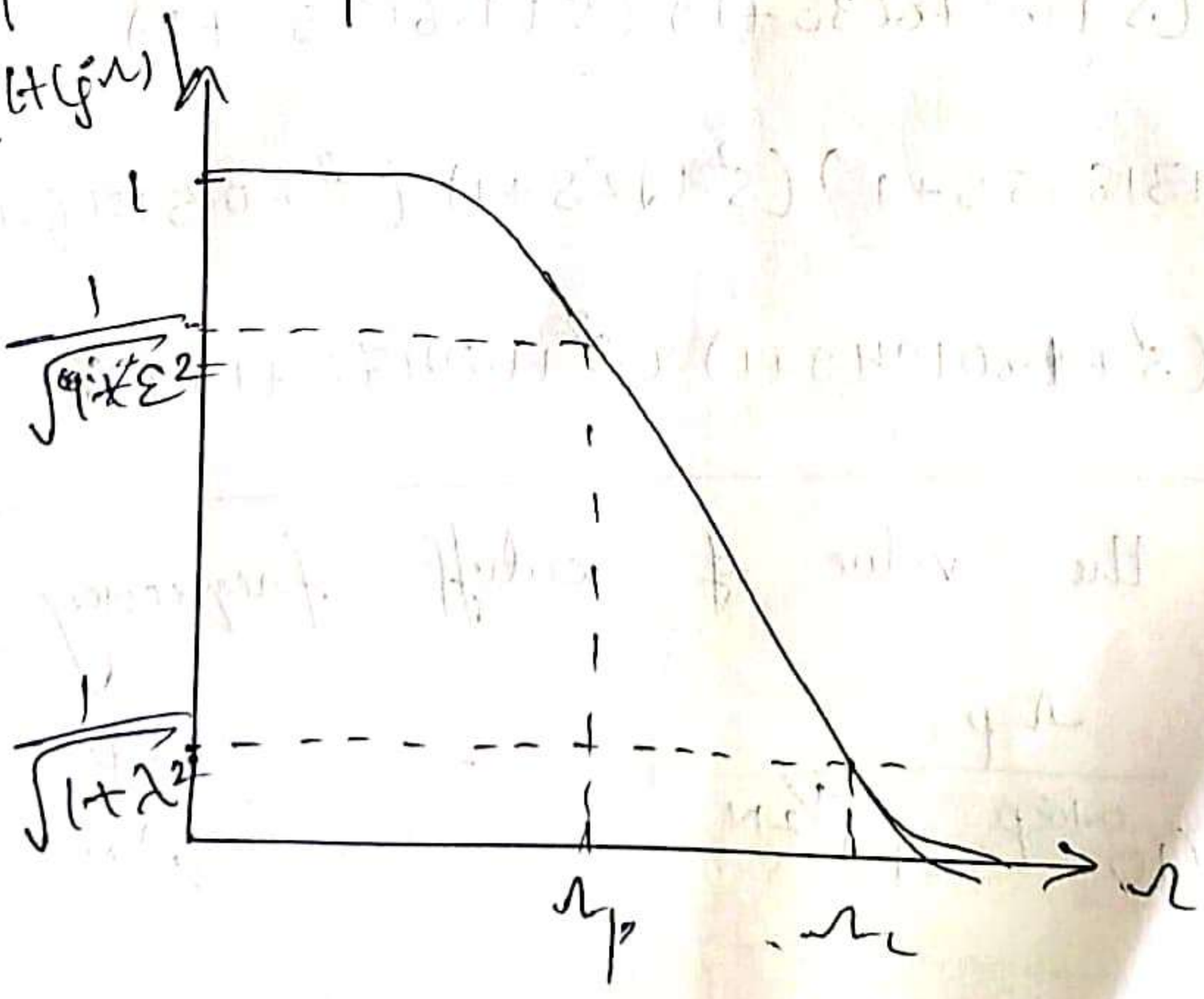
1. From the given specifications find the order of the filter "N"

$$N \geq \frac{\log \sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log\left(\Omega_s/\Omega_p\right)}$$

- where α_s – Stopband attenuation in dB.
  α_p – Passband attenuation in dB.
  Ω_p – Passband frequency in rad/sec.
  Ω_s – Stopband frequency in rad/sec.

(2)

Graphical representation of order



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$$N \geq \frac{\log\left(\frac{\lambda}{\varepsilon}\right)}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$\lambda = \left(10^{0.1\alpha_s} - 1\right)^{1/2}$$

$$\varepsilon = \left(10^{0.1\alpha_p} - 1\right)^{1/2}$$

(2) Round off it to the next higher integer.

(3) Find the transfer function H(S) for $\Omega_c = 1$ rad/sec, for the value of "N".

N	Denominator of H(S).
1	$S+1$
2	$s^2 + \sqrt{2}S + 1$
3	$(S+1)\quad (s^2+S+1)$
4	$(s^2+0.765378+1)\quad (s^2+1.84778+1)$
5	$(S+1)\ (s^2+0.6180338+1)\ (s^2+1.6180338+1)$
6.	$(s^2+1.9318558+1)\ (s^2+\sqrt{2}S+1)\ (s^2+0.5176458+1)$
7	$(S+1)\ (s^2+1.8019438+1)\ (s^2+1.2478+1)\ (s^2+0.4458+1)$

4. Calculate the value of cutoff frequency Ω_c

i) $\Omega_c = \frac{\Omega_p}{\left(10^{0.1\alpha_p} - 1\right)^{1/2N}} = \frac{\Omega_p}{\varepsilon^{1/N}}$

→ ii) $\Omega_c = \dfrac{\Omega_s}{(10^{0.1\alpha_s} - 1)^{1/2N}} = \dfrac{\Omega_s}{\lambda^{1/N}}$

5. Find the transfer function $H_a(s)$ for the above value of Ω_c by substituting $s = \frac{s}{\Omega_c}$ in $H(s)$.

6. For LPF design put $s \to \frac{s}{\Omega_c}$ in $H(s)$.,
   For HPF design put $s \to \frac{\Omega_c}{s}$ in $H(s)$.

* Design an analog Butterworth Filter that has a 2dB pass band attenuation at a frequency of 20 rad/sec, and at least 10dB stopband attenuation at 30 rad/sec.

Sol:- Given $\alpha_p = 2$ dB, $\Omega_p = 20$ rad/sec
$\alpha_s = 10$ dB $\Omega_s = 30$ rad/sec.

Step 1:- Order of the filter 'N'.

$$N \geq \frac{\log \sqrt{\dfrac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\log\left(\dfrac{\Omega_s}{\Omega_p}\right)}$$

$$N \geq \frac{\log \sqrt{[10^{0.1(10)} - 1] / [10^{0.1(2)} - 1]}}{\log [30/20]}$$

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$\therefore N \geq 3.37$

Step 2 :- Round off to the next higher integer.

$\therefore \boxed{N=4} \rightarrow$ order of the filter.

Step 3 :- $\therefore$ order of the filter $=4$. The numerator is always equal to 1. $\therefore$ The TF is

$$H(S) = \frac{1}{(S^2+0.76537S+1)(S^2+1.84775S+1)}$$

Step 4 :- cutoff frequency Ω_c

$\therefore \Omega_p < \Omega_S$

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p}-1)^{1/2N}} = \frac{20}{[10^{(0.1)(2)}-1]^{1/8}}$$

$\therefore \Omega_c = 21.3868$ rad/sec.

Step 5 :- In the question there is not given a filter is LPF (or) HPF, so we choose a LPF. design of LPF. put $S \rightarrow \frac{S}{\Omega_c}$

$$S \rightarrow \frac{S}{\Omega_c} = \frac{S}{21.3868}$$ in $H(S)$

$$\therefore H(S) = \frac{1}{\left[\left(\frac{S}{21.3868}\right)^2 + 0.76537\left(\frac{S}{21.3868}\right) + 1\right]} \quad *$$

$$\left[\left(\frac{S}{21.3868}\right)^2 + 1.8477\left(\frac{S}{21.3868}\right) + 1\right]$$

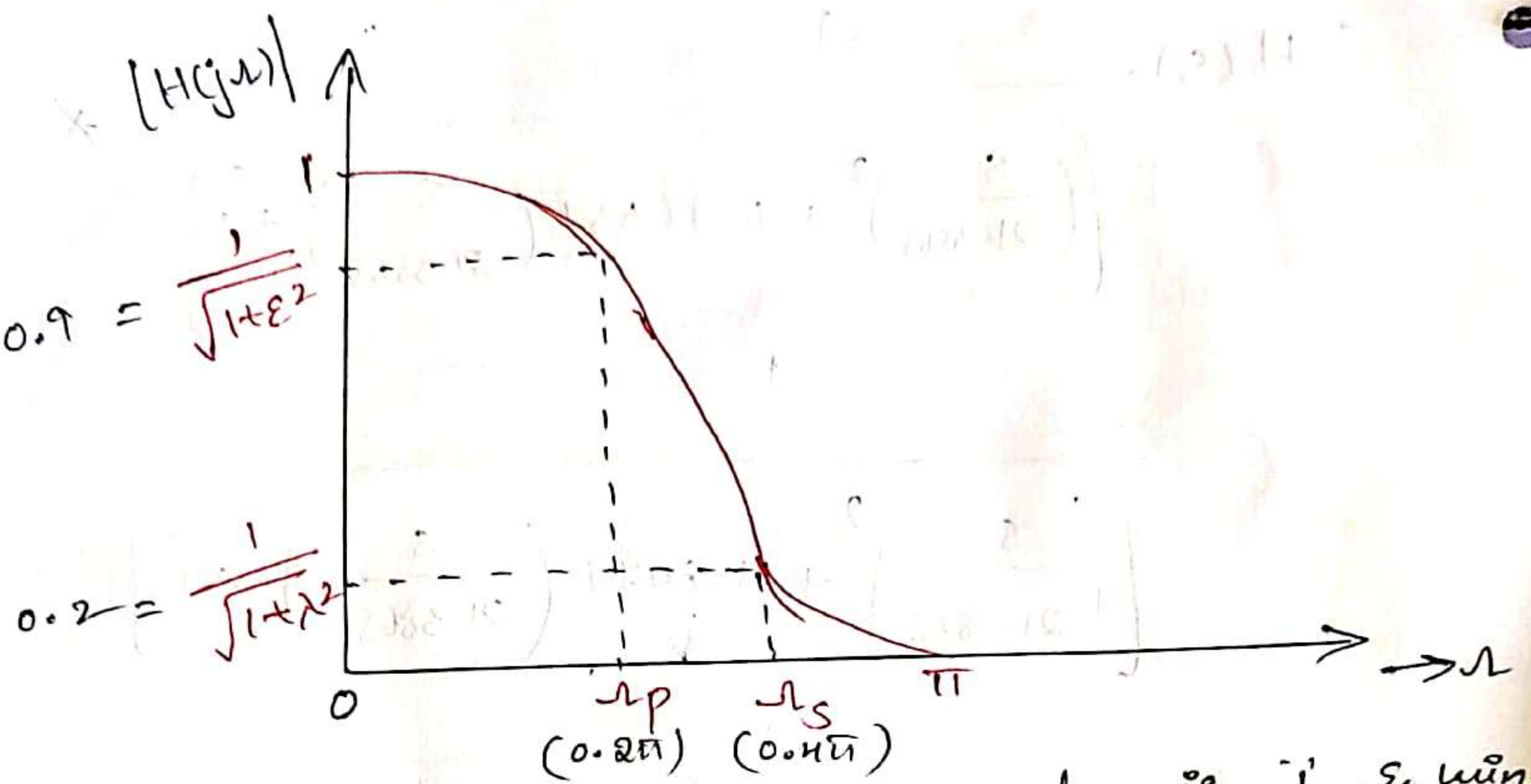
$$H(S) = \frac{1}{\left(\frac{S^2+16.3885+457.394}{(21.3868)^2}\right)\left(\frac{S^2+39.5165+457.394}{(21.3868)^2}\right)}$$

$$\therefore H(S) = \frac{20920.382}{(S^2+16.3685+457.394)(S^2+39.5165+457.394)}$$

(2) For the given specifications design an analog Butterworth Filter.

$0.9 \le |H(j\Omega)| \le 1$ for $0 \le \Omega \le 0.2\pi$

$|H(j\Omega)| \le 0.2$ for $0.4\pi \le \Omega < \pi$



From 0 to ωp the max value is '1' & min is 0.9.

$$\therefore \frac{1}{\sqrt{1+\varepsilon^2}} = 0.9$$

$$\sqrt{1+\varepsilon^2} = \frac{1}{0.9}$$

$$1+\varepsilon^2 = \left(\frac{1}{0.9}\right)^2$$

$$\varepsilon = \sqrt{\left(\frac{1}{0.9}\right)^2 - 1}$$

$$\boxed{\varepsilon = 0.484}$$

$$\frac{1}{\sqrt{1+\lambda^2}} = 0.2$$

$$\therefore \lambda = \sqrt{\left(\frac{1}{0.2}\right)^2 - 1} = 4.898$$

$\lambda_p = 0.2\pi$ rad/sec, $\lambda_s = 0.4\pi$ rad/sec.

Step-1 :- Order of the filter

$$N \geq \frac{\log\left(\frac{\lambda}{\varepsilon}\right)}{\log\left(\frac{\lambda_s}{\lambda_p}\right)}, \quad N \geq \frac{\log\left(\frac{4.898}{0.484}\right)}{\log\left(\frac{0.4\pi}{0.2\pi}\right)}$$

$$\therefore N \geq 3.34$$

Step 2 :- Rounding off it to the next higher integer

$N = 4$ → Order of the filter

Step 3 :- $\therefore N = 4$

$$H(S) = \frac{1}{(S^2+0.765375+1)(S^2+1.84775+1)}$$

Step 4 :- Calculate the cutoff frequency λ_c.

$\therefore \lambda_p < \lambda_s$

$$\therefore \lambda_c = \frac{\lambda_p}{(10^{0.1\alpha_p}-1)^{1/2N}} = \frac{0.2\pi}{(\varepsilon)^{1/N}}$$

$$\lambda_c = \frac{0.2\pi}{(0.484)^{1/4}} = 0.24\pi$$

Step 5 :- For a design of LPF specification

put $S = \frac{S}{\lambda_c} = \frac{S}{0.24\pi}$ in $H(S)$

$$\therefore H(S) = \frac{1}{\left[\left(\frac{S}{0.2\pi}\right)^2 + 0.76537\left(\frac{S}{0.2\pi}\right) + 1\right]\left[\left(\frac{S}{0.2\pi}\right)^2 + 1.8477\left(\frac{S}{0.2\pi}\right) + 1\right]}$$

$$\therefore H(S) = \frac{0.323}{(S^2 + 0.577S + 0.568)(S^2 + 1.393S + 0.568)}$$

=

(3) For the given specifications find the order of the Butterworth filter, $\alpha_p = 3dB$, $\alpha_S = 18dB$ $f_p = 1KHz$ & $f_S = 2KHz$.

Sol:- Given $\alpha_p = 3dB$, $\nu_p = 1KHz$

$\alpha_S = 18dB$, $\nu_S = 2KHz$

Step 1:- Order of the filter "N"

$$N \geq \frac{\log\sqrt{\frac{10^{0.1\alpha_S} - 1}{10^{0.1\alpha_p} - 1}}}{\log\left(\frac{\nu_S}{\nu_p}\right)}$$

$$N \geq \frac{\log\sqrt{[10^{0.1(18)} - 1]/[10^{0.1(3)} - 1]}}{\log\left[\frac{2\times1000}{1\times1000}\right]} \Rightarrow$$

(4) Design an analog Butterworth filter that has $\alpha_p = 0.5\,dB$, $\alpha_s = 22\,dB$, $f_p = 10\,KHz$, $f_s = 25\,KHz$.

Sol:- Given $\alpha_p = 0.5\,dB$, $\Omega_p = 10\,KHz$.

$\alpha_s = 22\,dB$, $\Omega_s = 25\,KHz$.

Order of the filter "N"

$$N \geq \frac{\log\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$N \geq \frac{\log\sqrt{\left[10^{0.1(22)}-1\right]\left[10^{0.1(0.5)}-1\right]}}{\log\left(\frac{25\times1000}{10\times1000}\right)}$$

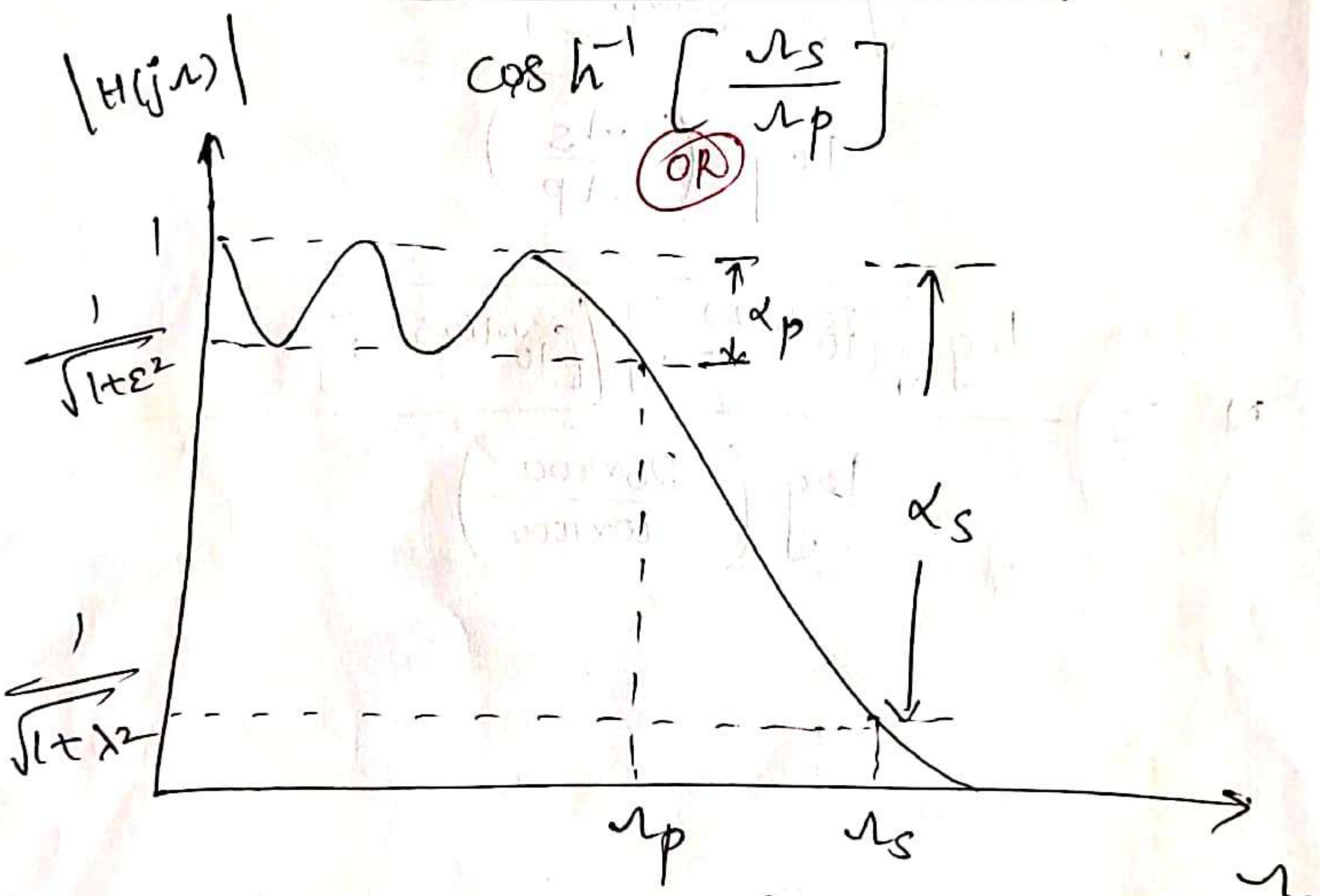
Analog Chebyshev Low Pass Filter:-

Steps to design chebyshev LPF:-

Step 1:- From the given specifications find the order of the filter "N"

$$N \geq \frac{\cosh^{-1}\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\cosh^{-1}\left[\frac{\Omega_s}{\Omega_p}\right]}$$

(OR)



$$N > \frac{\cosh^{-1}\left(\frac{\lambda}{\varepsilon}\right)}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)}$$

Step 2:- Rundoff it to the next higher integer.

Step 3:- Minor & Major axis of Ellipse

$$\log\left[\frac{\alpha \times 1000}{1 \times 1000}\right] \longrightarrow$$

(a) & (b)

$$a = \Omega_p \left[\frac{\mu^{1/N} - \mu^{-1/N}}{2} \right] \quad \& \quad b = \Omega_p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2} \right]$$

where $\mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^2}$

$$\varepsilon = \left(10^{0.1\alpha_p} - 1\right)^{1/2}$$

Step 4:- Poles of chebyshev filter. [ϕ_k is deg. value]

$$S_k = a\cos\phi_k + j\, b \sin\phi_k \qquad k = 1, 2, \ldots N$$

$$\phi_k = \frac{\pi}{2} + \left(\frac{2k-1}{2N}\right)\pi, \qquad k = 1, 2, \ldots N$$

Step 5:- Determinator polynomial determinator using the above poly.

Step 6:- Numerator of the transfer function depends on the value of N.

i) N = Odd, put s=0 in the denominator polynomial which is the numerator.

ii) N = even, put s=0 in the denominator polynomial and divide by $\sqrt{1+\varepsilon^2}$, which is the numerator.

(1) Design the a chebyshev filter for the given specifications $\alpha_p = 3dB$, $\alpha_s = 16\,dB$, $f_p = 1KHz$ and $f_s = 2KHz$.

Sol:- Given $f_p = 1KHz$ convert this into rad/sec.

i.e. multiply with $2\pi \Rightarrow 1KHz \times 2\pi$

$1000 \times 2\pi = 2000\pi$ rad/sec.

$\therefore f_p = 1KHz \Rightarrow \Omega_p = 1000 \times 2\pi = 2000\pi$ rad/sec

$f_s = 2KHz \Rightarrow \Omega_s = 2000 \times 2\pi = 4000\pi$ rad/sec

$\alpha_p = 3dB$; $\alpha_s = 16dB$.

Step 1:-

$$N \geq \frac{\cosh^{-1}\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$N \geq \frac{\cosh^{-1}\sqrt{\frac{10^{0.1(16)}-1}{10^{0.1(3)}-1}}}{\cosh^{-1}\left(\frac{4000\pi}{2000\pi}\right)}$$

$\therefore N \geq 1.91$.

Step 2: Order of the filter is $N = 2$.

Step 3:- Minor and Major axis of the Ellipse

$$a = \Omega_p\left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right] \quad \& \quad b = \Omega_p\left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right]$$

$\omega_3 \left[\frac{\quad}{1\times1000}\right]$

$\therefore \mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^2}$ and

$\varepsilon = (10^{0.1\alpha p} - 1)^{1/2} = (10^{0.1(3)} - 1)^{1/2} = 1$

$\therefore \mu = 2.414$

after substituting the 'μ' value and ω_p value $a = 910\pi$ and $b = 2197\pi$.

don't substitute the value of $\pi = 3.14$ like just kept the calculator in rad/sec.

Step 4:- poles.

$$s_k = a\cos\phi_k + j\, b \sin\phi_k, \quad k = 1, 2, \ldots N$$

$$\phi_k = \frac{\pi}{2} + \left(\frac{2k-1}{2N}\right)\pi, \quad k = 1, 2, \ldots N$$

$$k=1, \phi_1 = \frac{\pi}{2} + \left(\frac{2(1)-1}{2(2)}\right)\pi, \quad k = 1, 2, \ldots N$$

$$= \frac{\pi}{2} + \frac{\pi}{4} = 135^0$$

$$k=2, \phi_2 = \frac{\pi}{2} + \left(\frac{2(2)-1}{2(2)}\right)\pi$$

$$= \frac{\pi}{2} + \frac{3\pi}{4} = 225^0$$

$\therefore s_1 = 910\pi \cos(135) + j\, 2197\pi \sin(135^0)$

$s_1 = -643.46\pi + j\, 1554\pi$

don't type any deg's here type just normal value.

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after that press the calc option then type the 135.

For this calculations first set the calculator in "complex mode" after that degree mode" then type the value.

S_2 2 $910\angle$ cos (225) + j $2194\angle$ Sin (225)

$S_2 = -643.46\angle - j\,1554\angle$

Step 5:-

Denominator of H(s).

$\{(a-b)(a+b) = 2a^2+b^2\}$

$\Rightarrow (s+643.46\angle - j\,1554\angle)(s+643.46\angle + j\,1554\angle)$.

$\Rightarrow (s+643.46\angle)^2 + (1554\angle)^2$

Step 6:- Numerator of H(s).

$\therefore N=2$, put $s=0$. in the denominator. &

divide by $\sqrt{1+\varepsilon^2} \Rightarrow \dfrac{(643\angle-46\angle)^2+(1554\angle)^2}{\sqrt{1+1}}$

$= 1414.38\angle$

$\therefore H(2) = \dfrac{(1414.38\angle)^2}{(s+643.46\angle)^2+(1554\angle)^2}$

Design of IIR Filters from Analog Filters:-

There are several methods that can be used to design digital filters having an infinite duration unit sample response.

1. Approximation of derivatives.
2. The impulse invariant transform
3. The bilinear transformation.
4. The matched Z-Transformation technique.

INDIRECT METHOD:-

To design an analog prototype filter for the given specification:

To transform the analog prototype to digital filter.

① Map digital filter specifications into those for an equivalent analog filters.

Analog ⟷ Digital

Ω_p ω_p → Passband frequency

Ω_s ω_s → Stopband frequency

Ω_c ω_c → cutoff frequency.

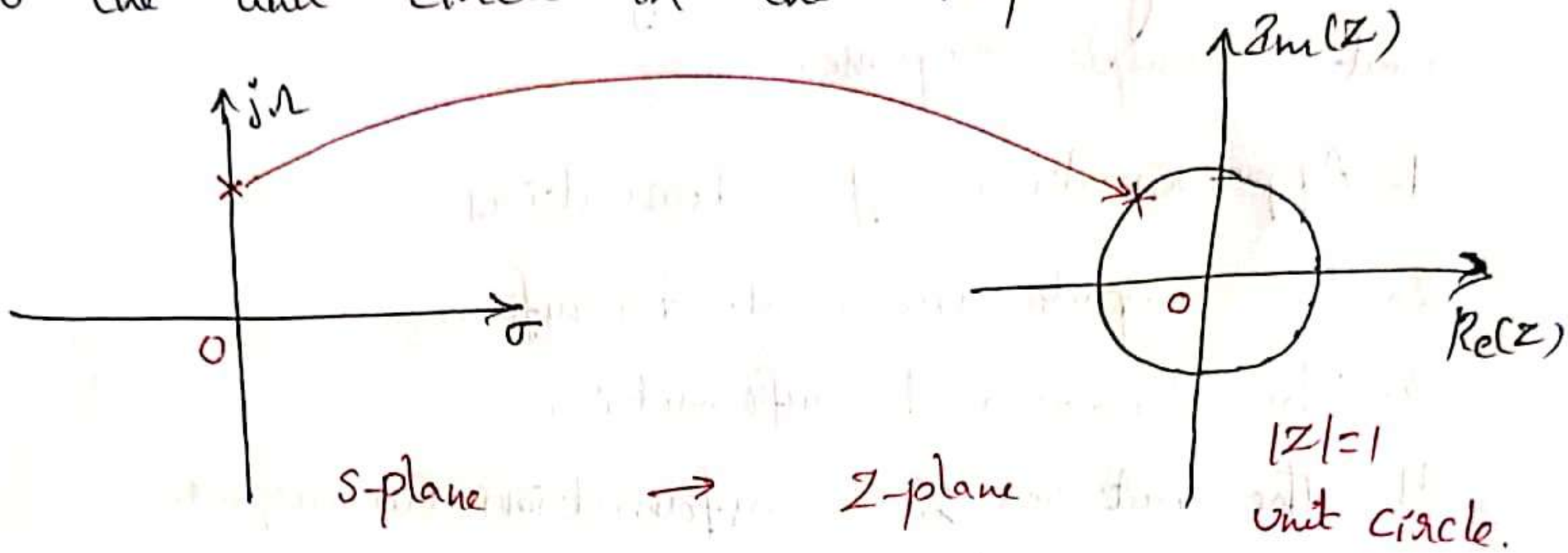
② Derive the analog transfer function H(S).

③ Transform the analog transfer function into an equivalent digital filter transfer function.

$H(S) \rightarrow H(Z)$; $|H(j\Omega)| = |H(e^{j\omega})|$.

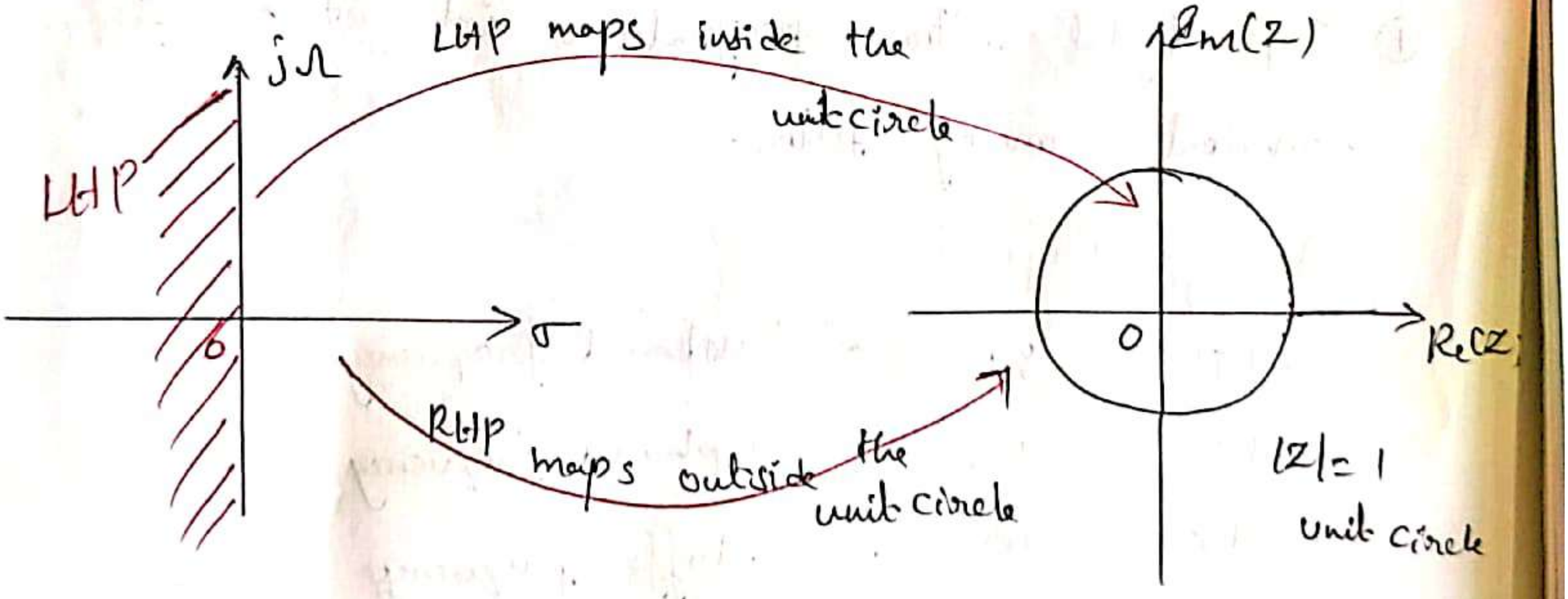
Desirable Properties in the transfer of S-plane to Z-plane.

① The "jΩ" axis in the S-plane should map in to the unit circle in the Z-plane.



② There will be a direct relationship between the two frequency variables (Ω, ω) in two domains.

② The left half plane of the S-plane should map into an inside the unit circle in Z-plane.



Thus a stable Analog filter can be converted into a stable digital filter.

Impulse Invariant Method:-

The unit Impulse response h(n) of the digital filter can be obtained by Sampling the impulse response of analog filter

$$h(n) = h_a(nT) ; \quad [H_a(t)|_{t=nT}]$$

↙ Impulse response ↘ sampled version of Analog filter.

Design steps:-

(1) Find the transfer function of Analog Filter $H_a(s)$

(2) Select the sampling rate of the Digital filter

$T \rightarrow$ sec per sample.

(3) Express the Analog filter transfer function as the sum of single pole Filter.

$$H_a(s) = \sum_{k=1}^{N} \frac{C_k}{s - P_k}$$

where $\{C_k\} \rightarrow$ coefficient of the partial fraction expansion.

$\{P_k\} \rightarrow$ Poles of the analog filter.

$N \rightarrow$ Order of the filter.

(4) Find the Z-Transform of the Digital filter

$$H(z) = \sum_{k=1}^{N} \frac{C_k}{1 - e^{P_k T} z^{-1}}$$

For high sampling rate we have to multiply the numerator with sampling period.

$$H(z) = \sum_{k=1}^{N} \frac{T \cdot C_k}{1 - e^{P_k \cdot T} z^{-1}} \quad ; \quad T >> 1.$$

* Determine the $H(z)$ for the analog Transfer function $H(s) = \frac{2}{(s+1)(s+2)}$, using Impulse Invariant method and assume here $T = 1 sec$.

Sol:- Step 1: Given Transfer function (Analog)

$$H(s) \text{ (or) } Ha(s) = \frac{2}{(s+1)(s+2)}$$

step 2:- The sampling period $T = 1 sec$.

step 3:- Express $H(s)$ as the sum of single-pole filter.

$$H(s) = \sum_{k=1}^{N} \frac{C_k}{s - P_k}$$

where $C_k \rightarrow$ coefficient of partial fraction expansion

$P_k \rightarrow$ Poles of filter.

$$H(s) = \frac{2}{(s+1)(s+2)} \Rightarrow \frac{A}{s+1} + \frac{B}{s+2}$$

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Find the coefficients using Partial fraction Expansion.

$$A \longrightarrow (s+1)\frac{2}{(s+1)(s+2)}\Big|_{s=-1}$$

$$A = \frac{2}{-1+2} \Rightarrow \boxed{A = 2}$$

$$B \longrightarrow (s+2)\cdot\frac{2}{(s+1)(s+2)}\Big|_{s=-2}$$

$$B = \frac{2}{-2+1} \Rightarrow \boxed{B = -2}$$

$\therefore \sum_{k=1}^{N} \frac{C_k}{s-P_k}$ here a & b are C_k values (coefficients)

$$\therefore H(s) = \frac{2}{s-P_1} - \frac{2}{s-P_2}$$

$$= \frac{2}{s-(-1)} - \frac{2}{s-(-2)}$$

($P_1 = -1$, $P_2 = -2$)

Step 4:- Compute the Z-Transform of digital filter $H(z) = \sum_{k=1}^{N} \frac{C_k}{1-e^{P_k T} z^{-1}}$

$$\Rightarrow \frac{C_1}{1-e^{P_1 T} z^{-1}} + \frac{C_2}{1-e^{P_2 T} z^{-1}}$$

$\therefore P_1 = -1$, $P_2 = -2$, $T = 1$, $C_1 = 2$, & $C_2 = -2$

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Substitute the all values in above eqn

$$H(z) = \frac{2}{1-e^{(-1)}z^{-1}} - \frac{2}{1-e^{-2}z^{-1}}$$

$$\therefore H(z) = \frac{2}{1-0.36782z^{-1}} - \frac{2}{1-0.13532z^{-1}}$$

For e values press shift & ln for calculator & enter the power values in brackets $e^{(-1)}$ or $e^{(-2)}$ etc.

$$H(z) = \frac{0.465z^{-1}}{1-0.503z^{-1}+0.04976z^{-2}}$$

$$H(z) = \frac{2(1-0.13532z^{-1}) - 2(1-0.36782z^{-1})}{(1-0.36782z^{-1})(1-0.13532z^{-1})}$$

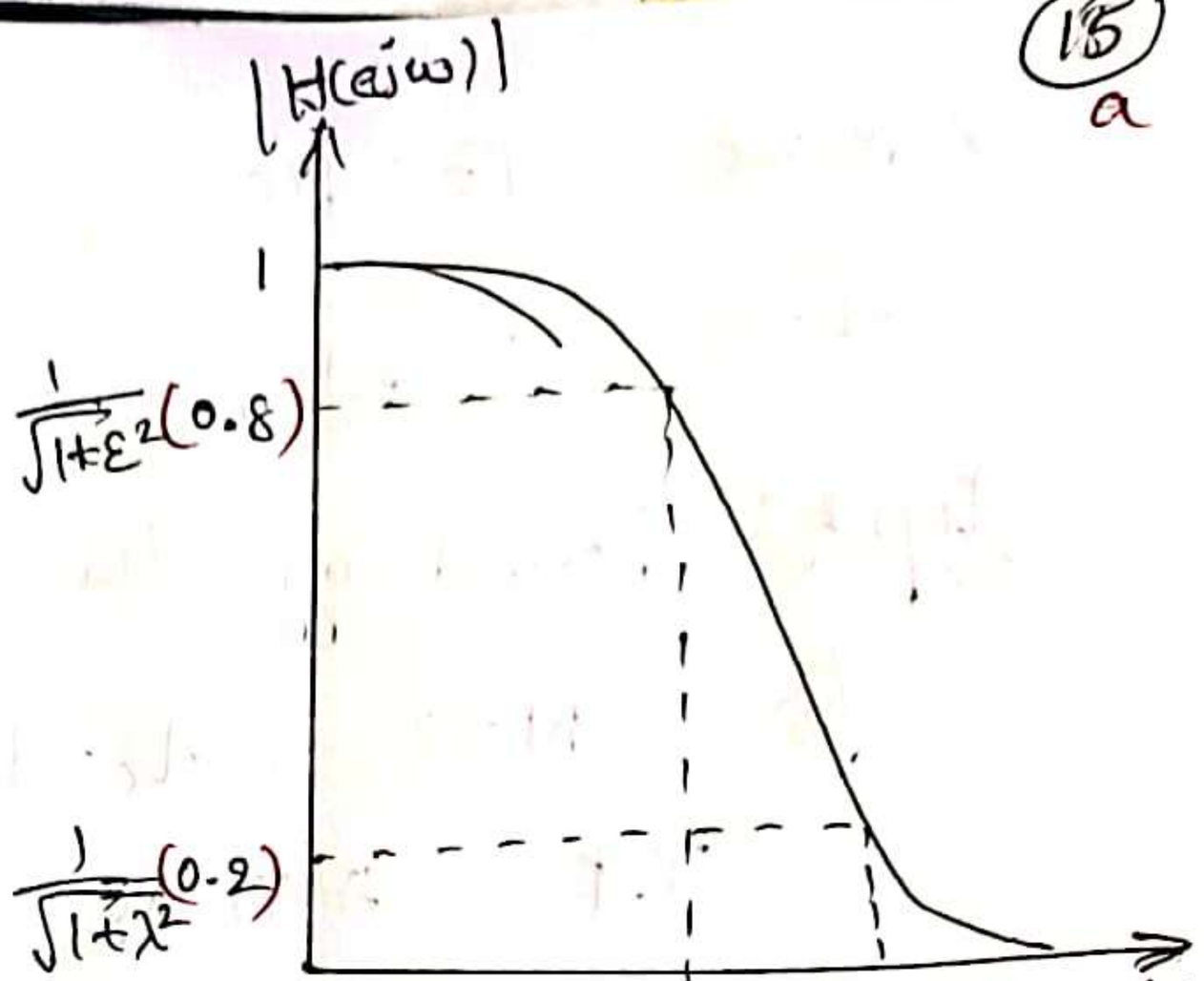
$$H(z) = \frac{2 - 0.27064z^{-1} - 2 + 0.73564z^{-1}}{1-0.13532z^{-1}-0.36782z^{-1}+0.04982z^{-2}}$$

$$\therefore H(z) = \frac{0.465z^{-1}}{1-0.5032z^{-1}+0.049762z^{-2}}$$

(2) Determine Design a BWF using the IIM for the given specification $0.8 \le |H(e^{j\omega})| \le 1$; $0 \le \omega \le \frac{\pi}{2}$

$|H(e^{j\omega})| \le 0.2$; $0.6\pi \le \omega \le \pi$.

a)



Sol:-

$\omega_p = 0.2\pi$

$$\frac{1}{\sqrt{1+\varepsilon^2}} = 0.8$$

$$\frac{1}{1+\varepsilon^2} = 0.64$$

$$0.64 + 0.64\varepsilon^2 = 1$$

$$0.64\varepsilon^2 = 1 - 0.64$$

$$\varepsilon^2 = \frac{0.36}{0.64}$$

$$\varepsilon^2 = 0.5625$$

$$\boxed{\varepsilon = 0.75}$$

$\omega_s = 0.6\pi$

$$\frac{1}{\sqrt{1+\lambda^2}} = 0.2$$

$$\frac{1}{1+\lambda^2} = 0.04$$

$$0.04 + 0.04\lambda^2 = 1$$

$$0.04\lambda^2 = 1 - 0.04$$

$$\lambda^2 = \frac{0.96}{0.04}$$

$$\lambda^2 = 24$$

$$\boxed{\lambda = 4.899}$$

Design Steps of IIR:-

Step 1:- Find the transfer function of Analog Butterworth filter $H_a(s)$.

Step 2:- Order of the filter

18

$$N \geq \frac{\log\left(\frac{\lambda}{\varepsilon}\right)}{\log\left(\frac{\omega_s}{\omega_p}\right)} = \frac{\log\left(\frac{4.899}{0.75}\right)}{\log\left(\frac{0.6\pi}{0.2\pi}\right)}$$

$$N \geq \frac{\log(6.532)}{\log(3)} \Rightarrow N \geq \frac{0.815}{0.477}$$

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Assume $T = 1$ sec.

$$\Omega_s = \frac{\omega_s}{T} = \omega_s. \qquad \Omega_p = \frac{\omega_p}{T} = \omega_p$$

Step 2:- Round off the order $N = 2$.

for $N = 2$, $\Omega_c = 1$. the transfer function

T.F, $H(s)$ is $H(s) = \dfrac{1}{s^2 + \sqrt{2}s + 1}$

Step 3:- Calculate the Ω_c

$$\Omega_c = \frac{\Omega_p}{\varepsilon^{1/N}} = \frac{0.2\pi}{(0.75)^{1/2}} = \frac{0.25}{0.866} = 0.7255$$

$$\boxed{\Omega_c = 0.7255}$$

Step 4:- The Transfer function of an analog Butter worth filter with $\Omega_c = 0.7255$

$$H_a(s) = H(s)\Big|_{s = \frac{s}{\Omega_c} = \frac{s}{0.7255}}$$

$$H_a(s) = \frac{1}{\left(\frac{s}{0.725}\right)^2 + \sqrt{2}\left(\frac{s}{0.725}\right) + 1}$$

$$= \frac{1}{\frac{s^2}{0.526} + 1.9495 + 1}$$

$$H_a(s) = \frac{0.526}{s^2 + 1.035 + 0.526}$$

Sampling rate $T = 1\text{sec}$.

(5) Express the $H_a(S)$ as the sum of single pole filter, $H_a(S) = \sum_{K=1}^{M} \frac{C_K}{S - P_K}$

$$N=2, \quad H_a(S) = \frac{C_1}{S-P_1} + \frac{C_2}{S-P_2}$$

$$S^2 + 1.03S + 0.526$$

$$S = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-1.03 \pm \sqrt{(1.03)^2 - 4(1)(0.526)}}{2(1)}$$

$$= \frac{-1.03 \pm \sqrt{1.0609 - 2.104}}{2}$$

$$= \frac{-1.03 \pm \sqrt{-1.0431}}{2} \Rightarrow \frac{-1.03 \pm j\,1.02}{2}$$

$$\boxed{S = -0.51 \pm j0.51}$$

$P_1 = -0.51 - j0.51$ & $P_2 = -0.51 + j0.51$

$$H_a(S) = \frac{0.526}{(S + 0.51 + j0.51)(S + 0.51 - j0.51)}$$

Using Partial Fraction Expansion method.

$$Ha(S) = \frac{C_1}{(S+0.51+j0.51)} + \frac{C_2}{(S+0.51-j0.51)}$$

C_1 :

$$C_1 = \frac{0.526}{S+0.51+j0.51} \times S+0.51+j0.51 \Big|_{S=-0.51-j0.51}$$

$$\Rightarrow \frac{0.526}{-0.51-j0.51+0.51-j0.51}$$

$$C_1 = \frac{0.526}{-j1.02} \Rightarrow \boxed{0.516j = C_1}$$

$$C_2 = C_1^* = -0.516j$$

$$\boxed{C_1 = 0.516j \quad , C_2 = C_1^* = -0.516j}$$

$$Ha(S) = \frac{0.516j}{S+0.51+j0.51} - \frac{0.516j}{S+0.51-j0.51}$$

$$\boxed{Ha(S) = \frac{0.516j}{S-(-0.51)-j0.51)} - \frac{0.516j}{S-(-0.51+j0.51)}}$$

(6) Find the transfer function of the digital Butterworth filter.

$$H(Z) = \sum_{k=1}^{N} \frac{C_k}{1-e^{P_k T} \cdot Z^{-1}}$$

(15) C

$$N=2,\quad H(z)=\frac{C_1}{1-e^{P_1T}z^{-1}}+\frac{C_2}{1-e^{P_2T}z^{-1}}$$

$P_1=-0.51-j0.51$; $P_2=-0.51+j0.51$; $T=1\text{sec}$

$C_1=0.516j$; $C_2=-0.516j$

$$H(z)=\frac{0.516j}{1-e^{(-0.51-j0.51)(1)}z^{-1}}+\frac{(-0.516j)}{1-e^{(-0.51+j0.5)(1)}z^{-1}}$$

$$=\frac{0.516j}{1-\underbrace{e^{-0.51}}_{0.6}e^{-j0.51}z^{-1}}-\frac{0.516j}{1-\underbrace{e^{-0.51}}_{0.6}e^{+j0.51}z^{-1}}$$

$$H(z)=0.516j\left[\frac{1}{1-0.6e^{-j0.51}z^{-1}}-\frac{1}{1-0.6e^{+j0.51}z^{-1}}\right]$$

$$=0.516j\left[\frac{1-0.6e^{j0.51}z^{-1}-1+0.6e^{-j0.51}z^{-1}}{(1-0.6e^{j0.51}z^{-1})(1-0.6e^{j0.51}z^{-1})}\right]$$

$$=0.516j\left[\frac{0.6z^{-1}(-e^{j0.51}+e^{-j0.51})}{\left[1-0.6e^{j0.5}z^{-1}-0.6e^{-j0.51}z^{-1}+0.36e^{-j0.51}e^{j0.51}z^{-2}\right]}\right]$$

$$=0.516j\left[\frac{0.6z^{-1}(-2j\sin(0.51))}{1-0.6\left[e^{j2\pi}+e^{-j2\pi}\right]z^{-1}+0.36z^{-2}}\right]$$

$\left[\because e^{j\theta}+e^{-j\theta}=2\cos\theta \quad e^{j\theta}-e^{-j\theta}=2j\sin\theta.\right]$ $-e^{j\theta}+e^{-j\theta}=2j\sin\theta$

$$H(Z) = \frac{j0.2096 z^{-1} \quad (-2j \times 0.488)}{1 - 0.6(2 \cos 0.57) z^{-1} + 0.36 z^{-2}}$$

$$= \frac{-j^2 0.3021 z^{-1}}{1 - 0.6(2 \times 0.873) z^{-1} + 0.36 z^{-2}}$$

The transfer function of Butterworth digital filter

$$H(z) = \frac{0.3021 z^{-1}}{1 - 1.0476 z^{-1} + 0.36 z^{-2}}$$

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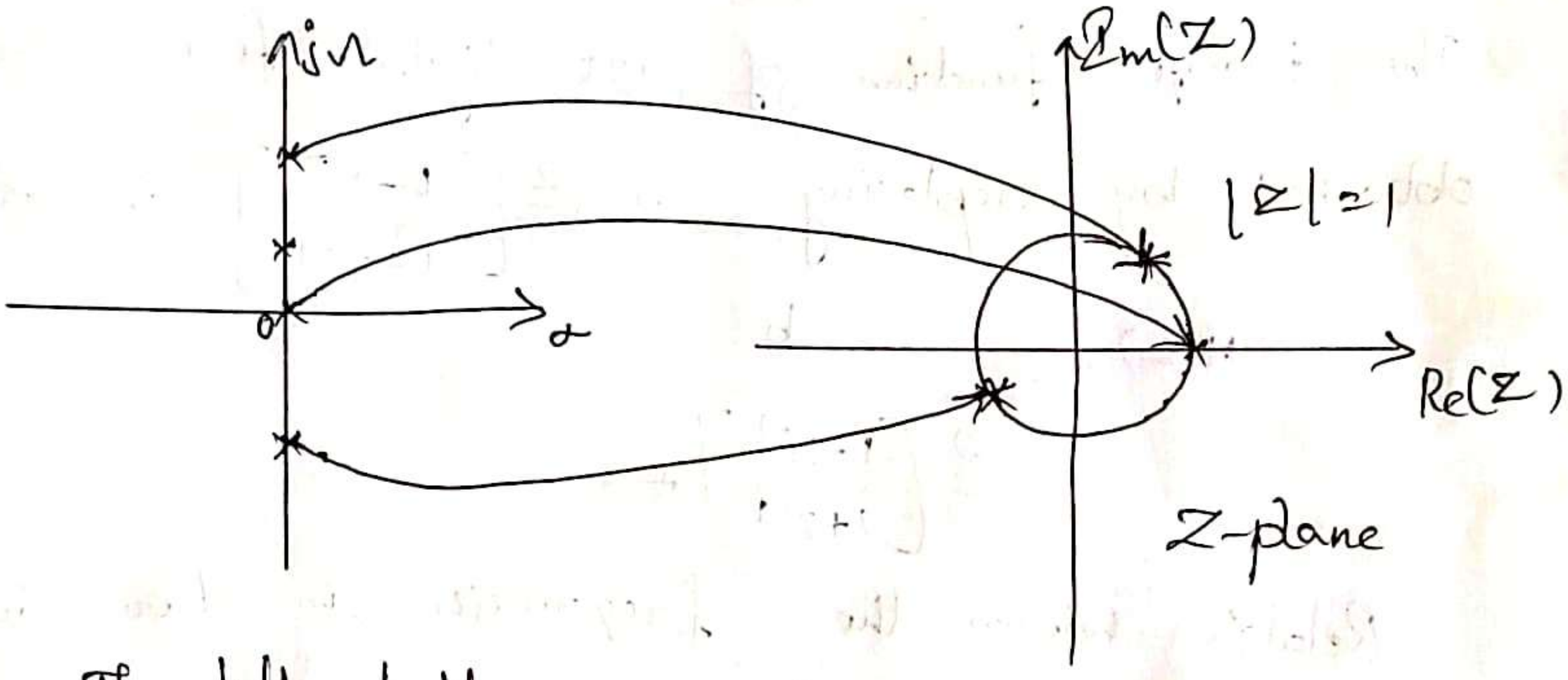
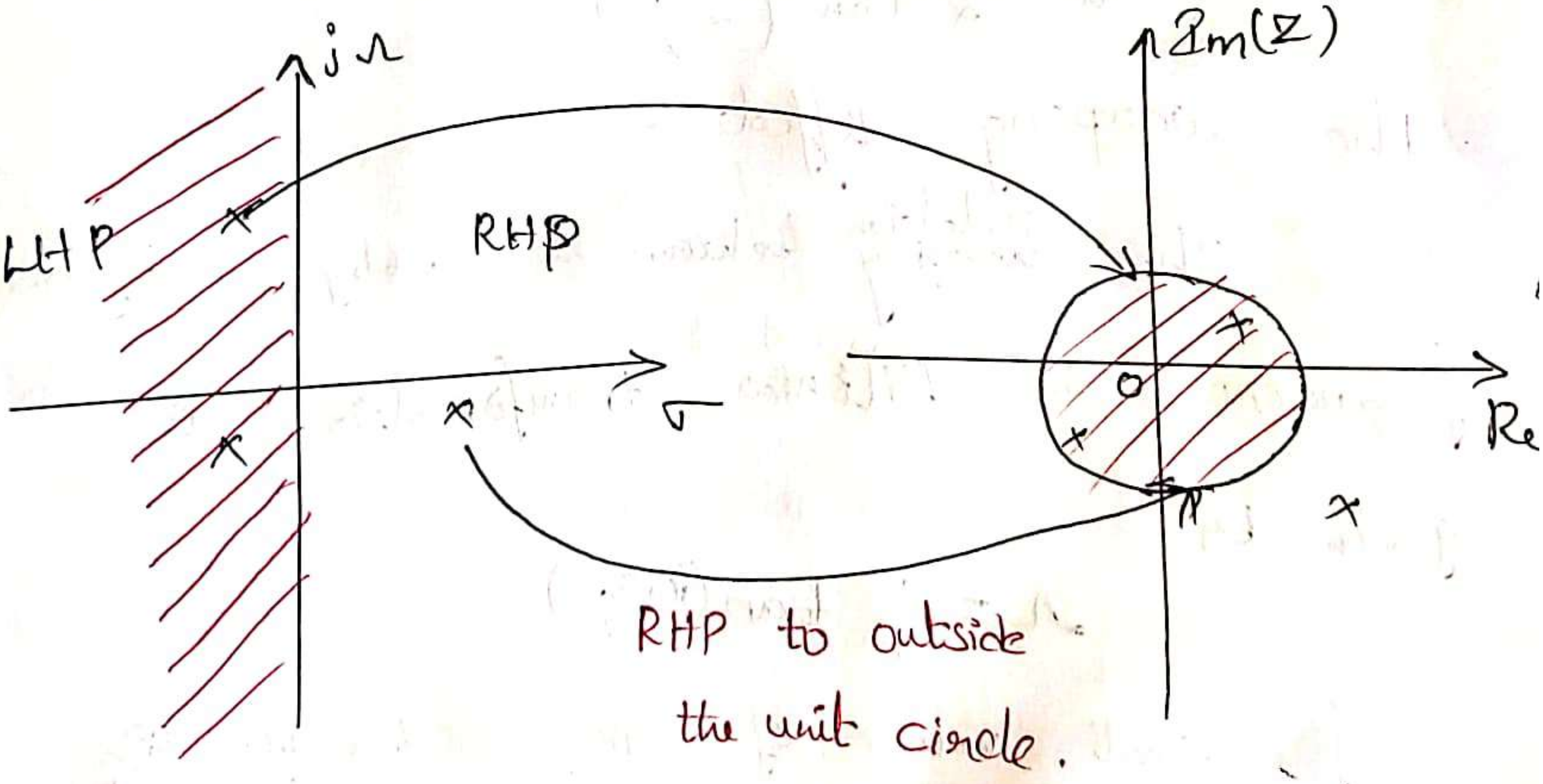
Bilinear Transformation:- It is a mapping method used to design IIR digital filter from an analog filter.

Properties:-

* The "jΩ" axis in the s-plane maps on to the unit circle in the z-plane.



The left half of s-plane maps into the inside the unit circle in the z-plane.



The right half of s-plane maps outside the unit circle in the z-plane.

Analysis:-

Let's consider an analog filter with the the transfer function.

$$H(S) = \frac{b}{s+a}$$

The transfer function of IIR digital filter can be obtained by replacing $S = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$ in $H(S)$.

$$H(z) = \frac{b}{\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right] + a}$$

Relation between the frequencies in two domains.

$$\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$$

$$\omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right)$$

The warping effect:-

The relation warping between the analog & digital frequencies in Bilinear transformation is given by

$$\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$$

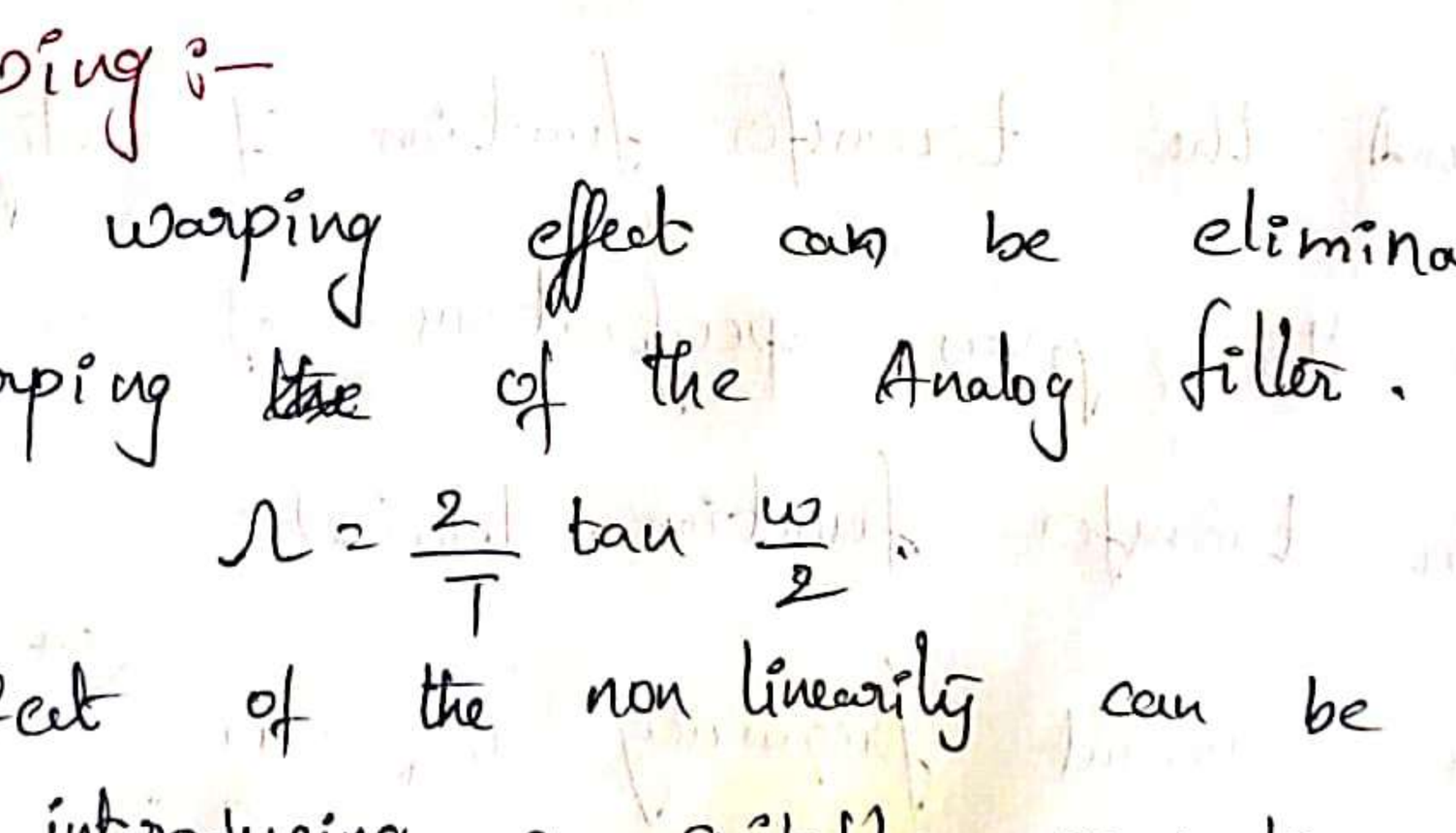
for small value of ω, $\tan\frac{\omega}{2} = \frac{\omega}{2}$

$$\Omega = \frac{2}{T} \times \frac{\omega}{2} = \frac{\omega}{T}$$

For low frequencies the relation btw "Ω" & "ω" is linear (i.e) the filter has the same amplitude response as that of analog filter.

For High frequencies, the relation btw "Ω" & "ω" is non-linear.

The effect of non-linearity at high frequenies introduces the distation in the response of the filter. This effect is known as warping effect.



Prewarping :-

The warping effect can be eliminated by prewarping of the Analog filter.

$$\Omega = \frac{2}{T} \tan \frac{\omega}{2}$$

The effect of the non linearity can be compensa-ted by introducing a suitable prescaling

prewarping of the critical frequencies to control the waping effect at high frequencies range.

Design steps of Bilinear Transformation Method:-

(1) From the given specifications find pre-warping analog frequencies. $\therefore \Omega = \frac{2}{T} \tan \omega/2$.

(2) Find H(S) of the analog filter with "Ω"

(3) Select the simpling rate of the digital filter "T".

(4) Find the transition transfor function of the digital filter.

$$H(z) = H(s)\Big|_{s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

=

P) Find the transfer function of digital filter for the given Specifications of an analog filter transfer function. $H_a(s) = \frac{s+0.1}{(s+0.1)^2 + 16}$

the resonant frequency $\omega_r = \frac{\pi}{2}$.

=

Q:- $H(S) = H(Z)\Big|_{S=\frac{2}{T}\left[\frac{1-Z^{-1}}{1+Z^{-1}}\right]}$

To obtain 'T' use transfer function $H_a(S) = \frac{S+9}{(S+9)^2+\Omega_c^2}$

$\therefore \Omega_c^2 = 16 \Rightarrow \Omega_c = 4$ rad/sec.

$\Omega = \frac{2}{T}\tan \omega/2$

Re arrange the equation.

$T = \frac{2}{\Omega_c}\tan\frac{\omega\Omega}{2}$

$T = \frac{2}{4}\tan\left[\frac{\pi/2}{2}\right]$ $\left\{\tan \pi/4 = 1\right.$

Put $\omega = \pi/2$ $T = \frac{2}{\pi}(1) = \boxed{T = 1/2}$

$\therefore H(Z) = H_a(S)\Big|_{S=\frac{2}{1}\left[\frac{1-Z^{-1}}{1+Z^{-1}}\right]}$

$H(Z) = \dfrac{S+0.1}{(S+0.1)^2+16}\Bigg|_{S=\frac{2}{1/2}\left[\frac{1-Z^{-1}}{1+Z^{-1}}\right]}$

$S = 4\left[\frac{1-Z^{-1}}{1+Z^{-1}}\right]$

$H(Z) = \dfrac{4\left[\frac{1-Z^{-1}}{1+Z^{-1}}\right]+0.1}{\left(4\left[\frac{1-Z^{-1}}{1+Z^{-1}}\right]+0.1\right]^2+16}$

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$$\frac{4(1-z^{-1}) + 0.1(1+z^{-1})}{(1+z^{-1})}$$

$$\therefore H(z) = \frac{(1+z^{-1})}{\left[\frac{4(1-z^{-1}) + 0.1(1+z^{-1})}{(1+z^{-1})}\right]^2 + 16(1+z^{-1})^2}$$

$$= \frac{}{}$$

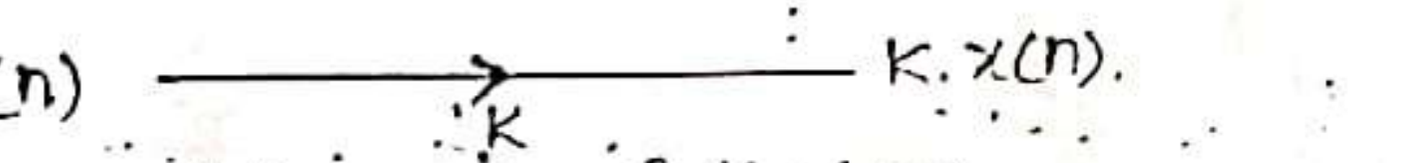
UNIT-3
INFINITE IMPULSE RESPONSE FILTER

Basic Structures of IIR Filters:-

1. Direct Form-I structure.
2. Direct Form-II structure.
3. Cascade Form structure.
4. parallel Form structure.

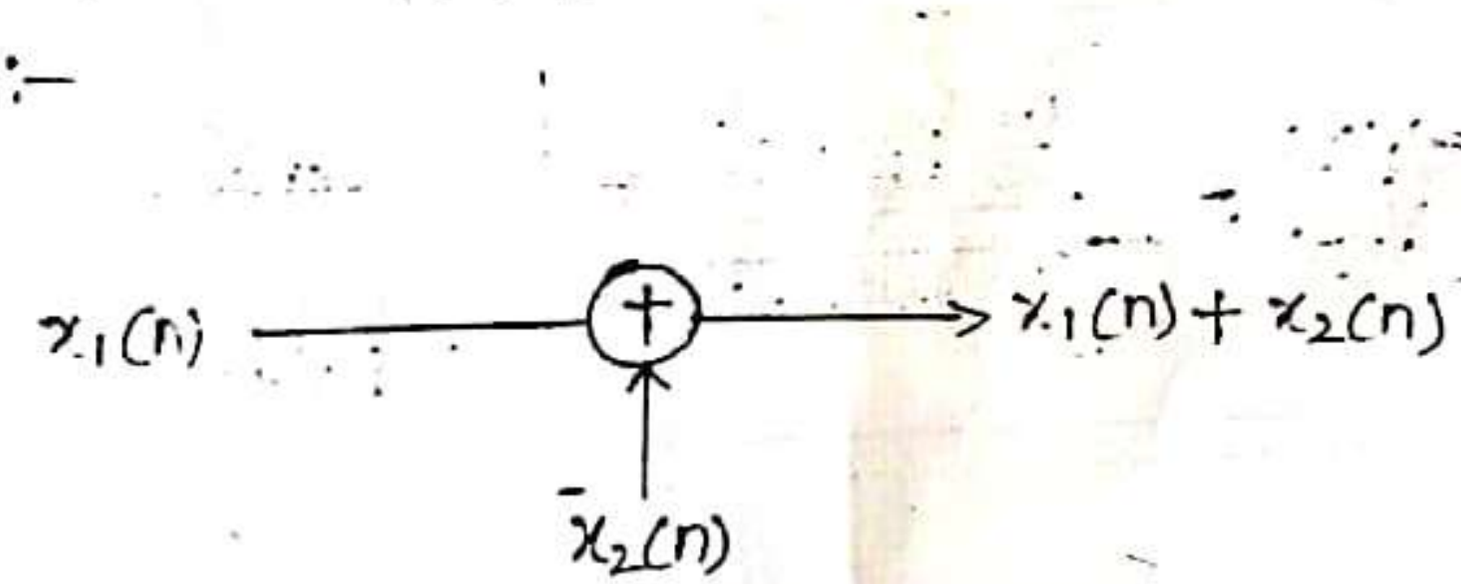
Basic Elements:-

1. Constant Multiplier:-

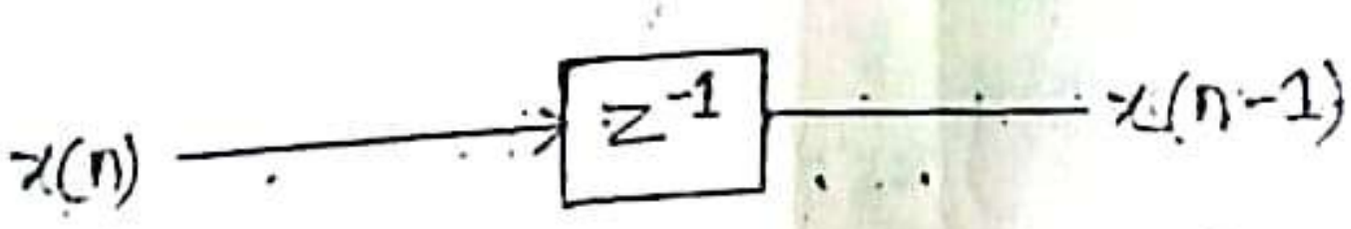


Where, K = 1, 2, 0.3, 0.7.

2. Adder:-



3. Delay Element:-



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Transfer Function of IIR System :-

In general, an IIR system is described by the differential equation,

$$y(n) = -\sum_{k=1}^{N} a_k \cdot y(n-k) + \sum_{k=0}^{M} b_k \cdot x(n-k) \longrightarrow (1)$$

Where,

a_k & b_k are constants.

Apply the z-Transform,

$$Y(z) = -\sum_{k=1}^{N} a_k z^{-k} \cdot Y(z) + \sum_{k=0}^{M} b_k z^{-k} \cdot X(z)$$

$$Y(z) + \sum_{k=1}^{N} a_k \cdot z^{-k} Y(z) = \sum_{k=0}^{M} b_k \cdot z^{-k} X(z)$$

$$Y(z)\left[1 + \sum_{k=1}^{N} a_k \cdot z^{-k}\right] = X(z)\left[\sum_{k=0}^{M} b_k \cdot z^{-k}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k \cdot z^{-k}}{1 + \sum_{k=1}^{N} a_k \cdot z^{-k}}$$

→ Zeroes

→ Poles

1. Direct Form-I Realization Structure :-

Consider an LTI recursive filter described by the differential equation,

$$y(n) = -\sum_{k=1}^{N} a_k \cdot y(n-k) + \sum_{k=0}^{M} b_k \cdot x(n-k) \longrightarrow (1)$$

$$y(n) = -[a_1 y(n-1) + a_2 y(n-2) + \ldots\ldots + a_N \cdot y(n-N) + b_0 x(n-0) + b_1 x(n-1) + \ldots\ldots + b_m \cdot x(N-M)]$$

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$$= -a_1 y(n-1) - a_2 y(n-2) - \ldots\ldots - a_N y(n-N) + b_0 x(n) + b_1 x(n-1) + \ldots\ldots + b_M x(n-M)$$

Let,

$$W(n) = b_0 x(n) + b_1 x(n-1) + \ldots\ldots + b_M x(n-M) \longrightarrow (1)$$

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) - \ldots\ldots - a_N y(n-N) + W_N \longrightarrow (2)$$

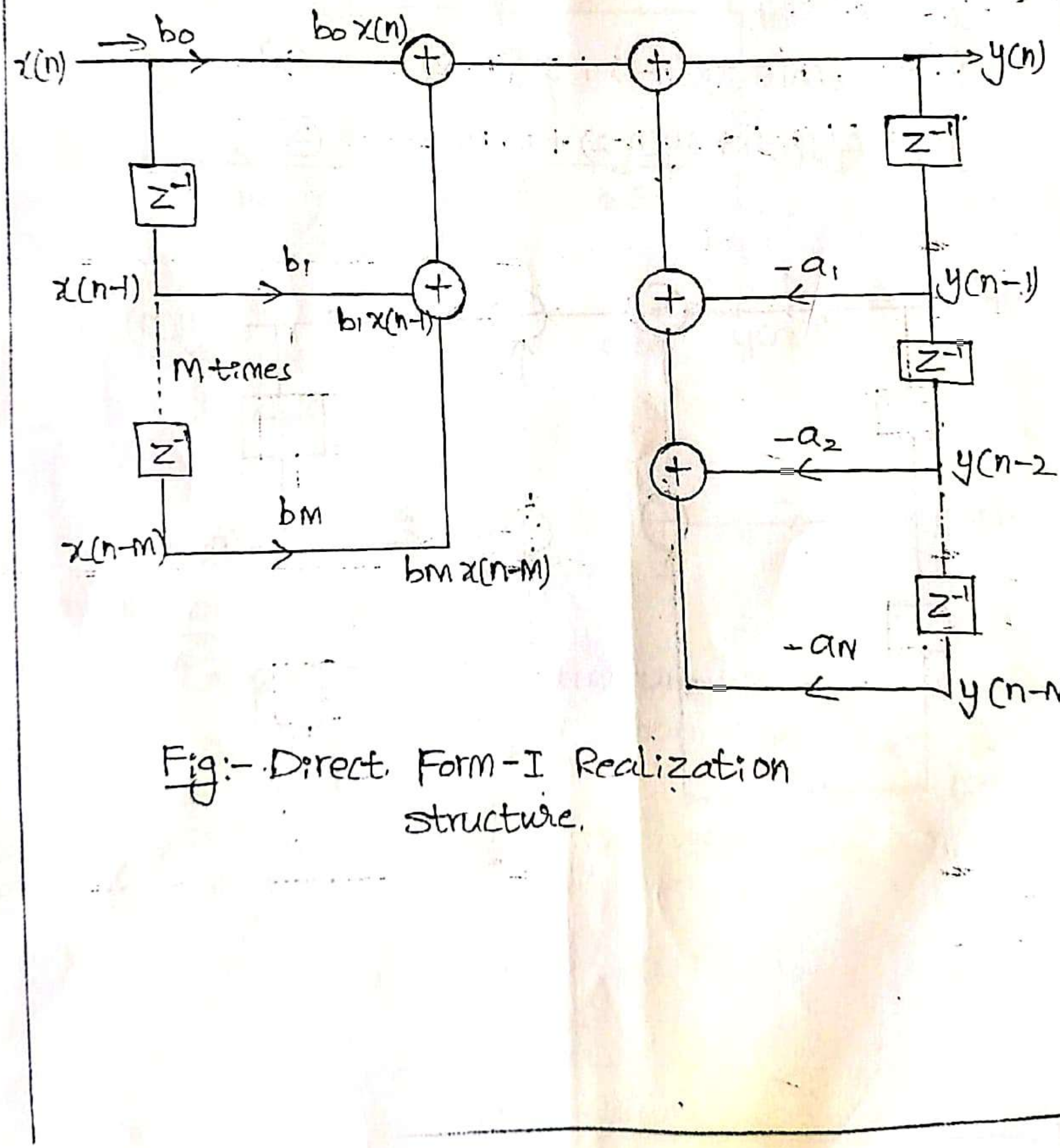

Fig:- Direct Form-I Realization structure.

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Numerical :-

Obtain the direct form one, realisation for the system described by the following difference equation.

i. $y(n) = 2y(n-1) + 3y(n-2) + x(n) + 2x(n-1) + 3x(n-2)$

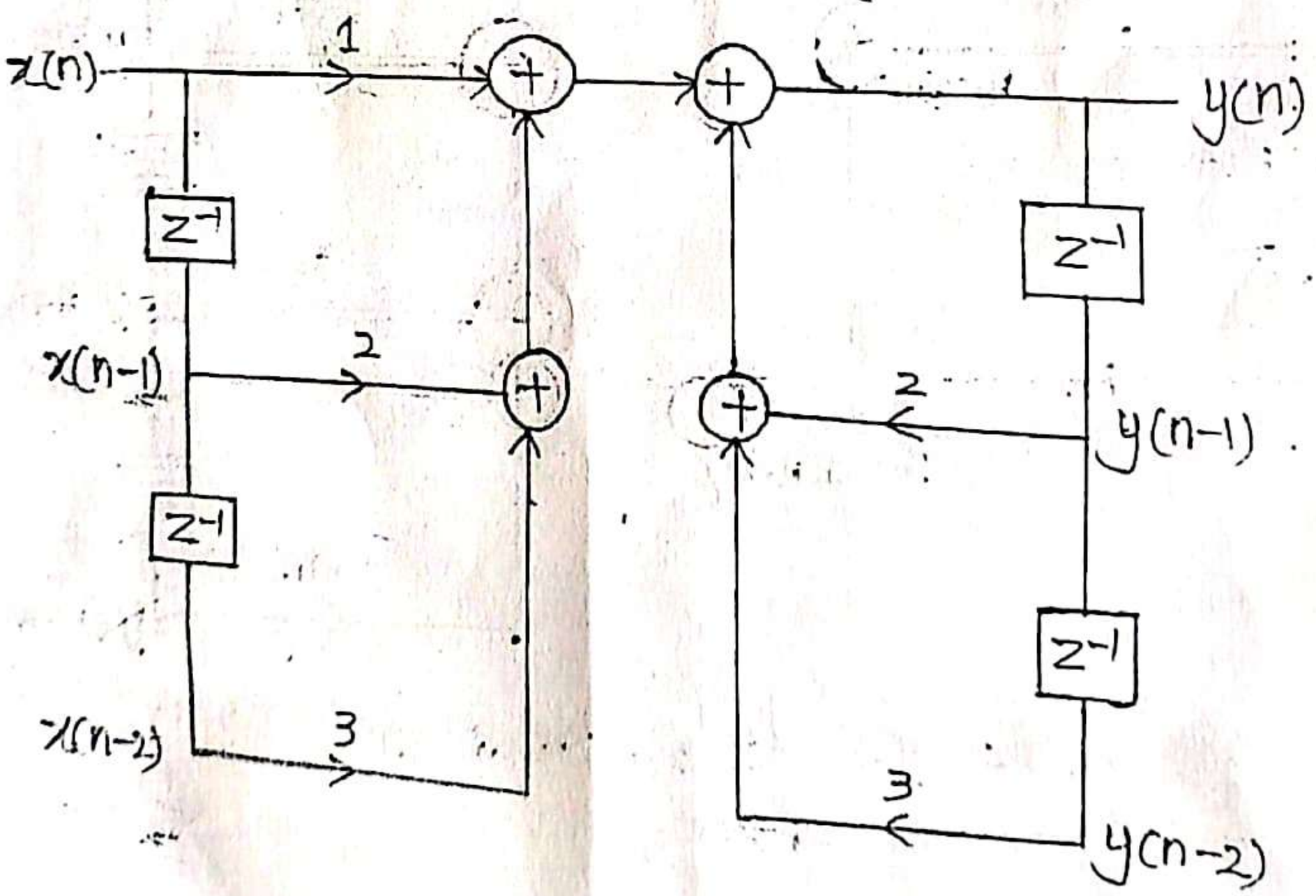
A. Given,

$$y(n) = 2y(n-1) + 3y(n-2) + x(n) + 2x(n-1) + 3x(n-2)$$

Let us consider,

$$W(n) = x(n) + 2x(n-1) + 3x(n-2) \longrightarrow (1)$$

$$y(n) = 2y(n-1) + 3y(n-2) + W(n) \longrightarrow (2)$$



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ii. $y(n) = 0.5\,y(n-1) + 0.06\,y(n-2) + 0.3\,x(n) + 0.5\,x(n-1)$.

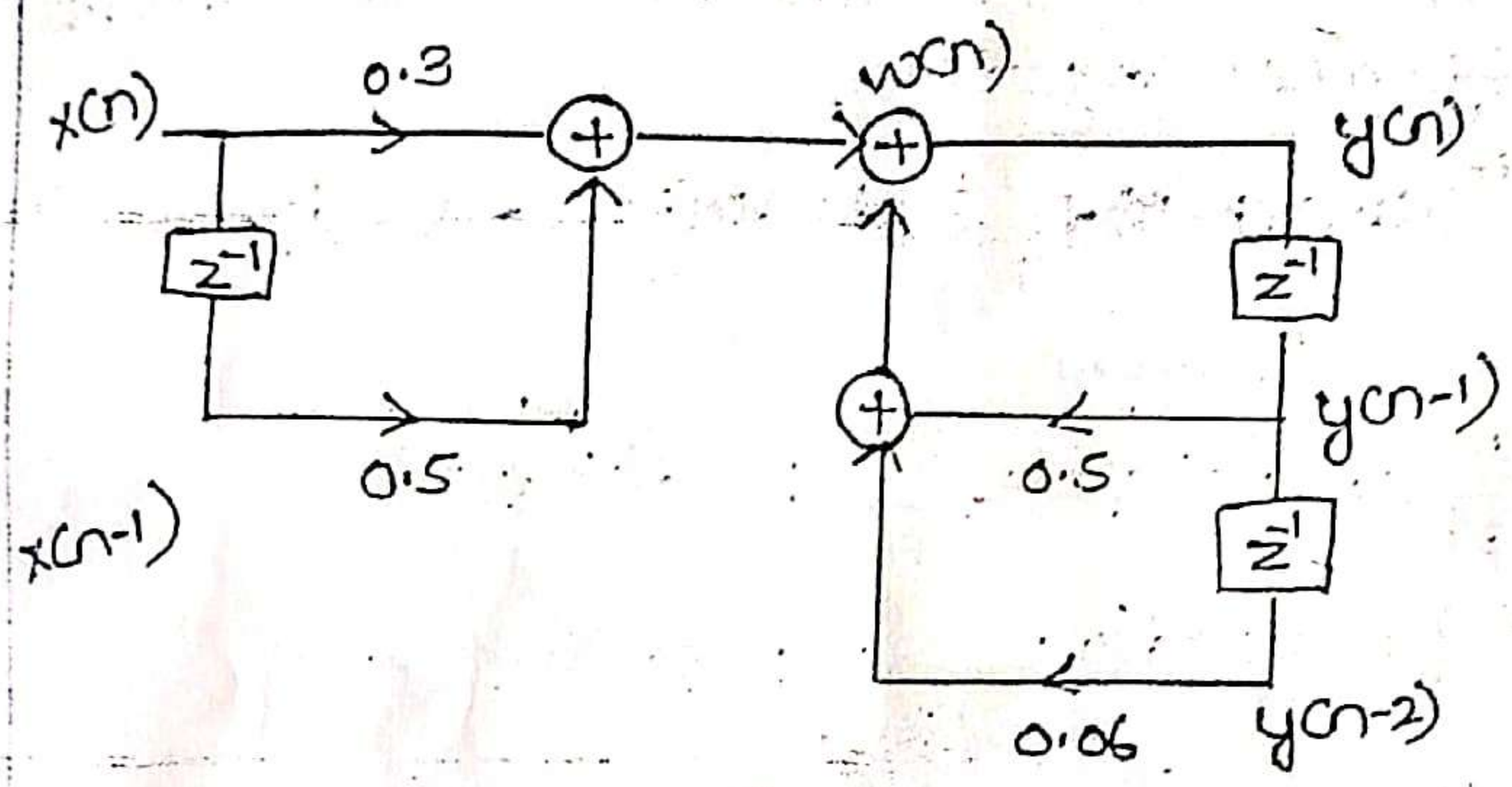
A. Given,

$$y(n) = 0.5\,y(n-1) + 0.06\,y(n-2) + 0.3\,x(n) + 0.5\,x(n-1)$$

Let,

$$w(n) = 0.3\,x(n) + 0.5\,x(n-1) \longrightarrow (1)$$

$$y(n) = 0.5\,y(n-1) + 0.06\,y(n-2) \longrightarrow (2)$$



Limitations:-

1. The Direct form-I Structure is having number of delay elements higher than the order of difference equation.
2. Hence it is a non-canonical structure.

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2. Direct Form-II Realisation structure:-

1. It is an alternative to the Direct Form-I realisation structure.

2. In this structure the number of delay elements used is equal to the order of the difference equation.

3. Let us consider an LTI recursive system described by the difference equation:

$$y(n) = -\sum_{k=1}^{N} a_k . y(n-k) + \sum_{k=0}^{M} b_k . x(n-k) \longrightarrow (1)$$

Apply the Z-Transform

$$Y(z) = -\sum_{k=1}^{N} a_k . z^{-k} Y(z) + \sum_{k=0}^{M} b_k . z^{-k} X(z)$$

$$Y(z) + \sum_{k=1}^{N} a_k . z^{-k} Y(z) = \sum_{k=0}^{M} b_k . z^{-k} X(z)$$

$$Y(z)\left[1 + \sum_{k=1}^{N} a_k . z^{-k}\right] = X(z)\left[\sum_{k=0}^{M} b_k . z^{-k}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k . z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$$\frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M . z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N . z^{-N}}$$

$$\frac{Y(z)}{W(z)} \times \frac{W(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}}{1} \times \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}}$$

$$\frac{Y(z)}{W(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}$$

$$Y(z) = b_0 W(z) + b_1 z^{-1} . W(z) + b_2 z^{-2} W(z) + \ldots + b_M z^{-M} W(z) \longrightarrow (2)$$

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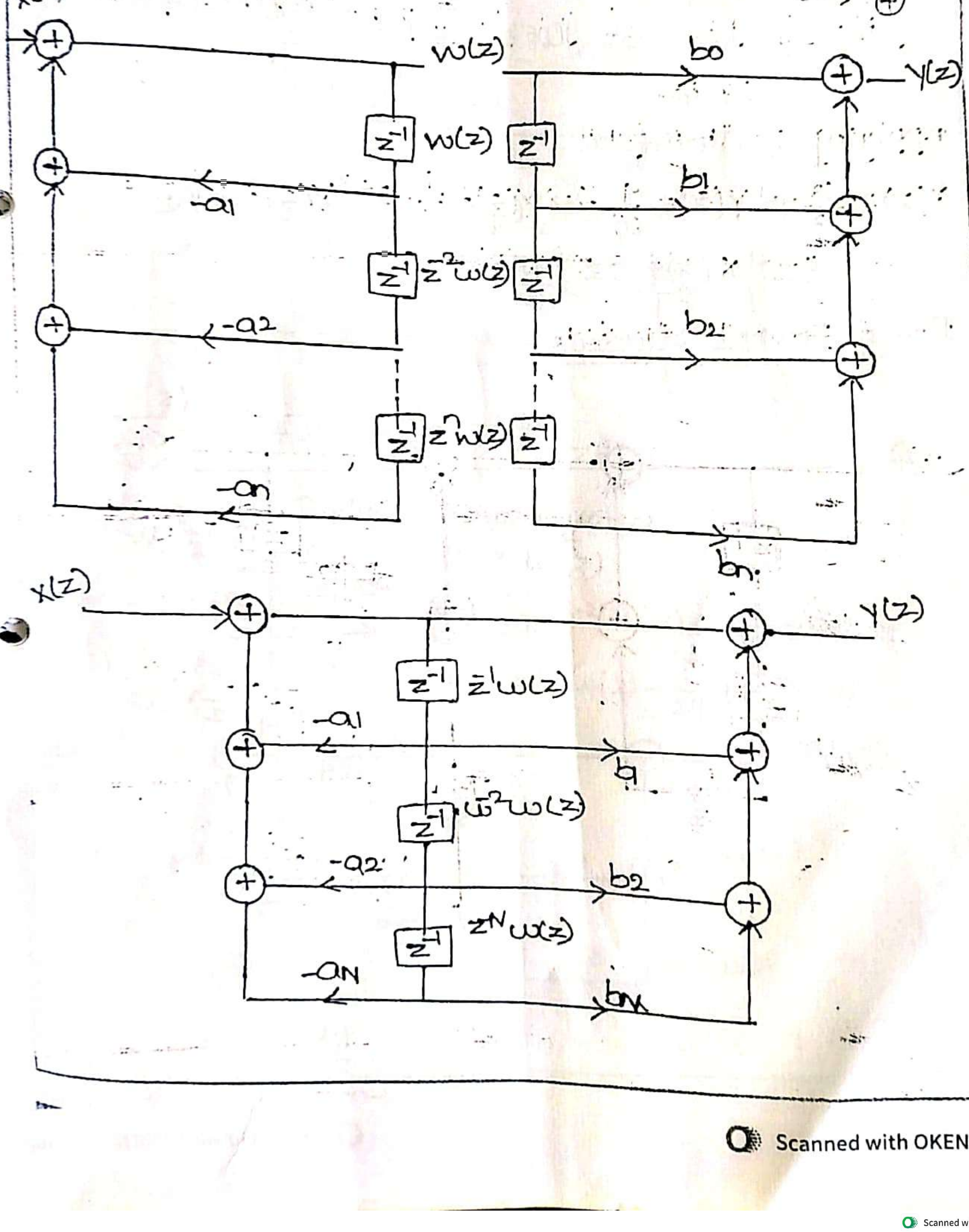
$$\frac{W(z)}{X(z)} = \frac{1}{1+a_1z^{-1}+a_2z^{-2}+\ldots+a_Nz^{-N}}$$

$$X(z) = W(z) + a_1z^{-1}W(z) + a_2z^{-2}W(z) + \ldots + a_Nz^{-N}W(z) \quad \rightarrow (3)$$

$$W(z) = X(z) - a_1z^{-1}W(z) - a_2z^{-2}W(z) - \ldots - a_Nz^{-N}W(z) \quad \rightarrow (4)$$



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11 Obtain the direct form-I & direct form-II realization of LTI system, described by the difference equation

$y(n) = -\frac{13}{12}y(n-1) - \frac{9}{24}y(n-2) - \frac{1}{24}y(n-3) + x(n) + 4x(n-1) + 3x(n-2)$

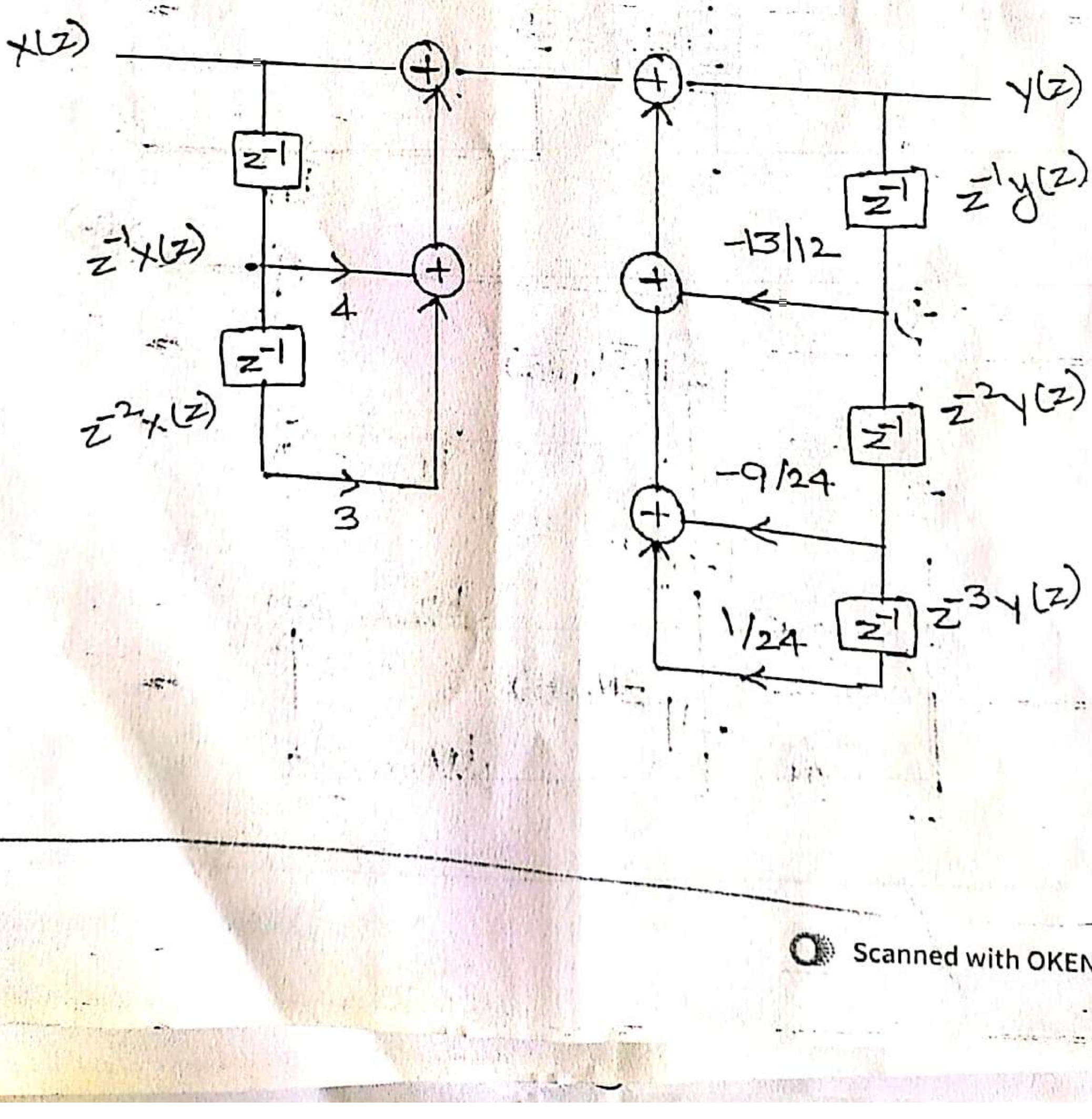
A. Given eq:

$y(n) = -\frac{13}{12}y(n-1) - \frac{9}{24}y(n-2) - \frac{1}{24}y(n-3) + x(n) + 4x(n-1) + 3x(n-2)$

Applying z-Transform,

$Y(z) = -\frac{13}{12}z^{-1}Y(z) - \frac{9}{24}z^{-2}Y(z) - \frac{1}{24}z^{-3}Y(z) + X(z) + 4z^{-1}X(z) + 3z^{-2}X(z)$

Direct Form-I Structure:-



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Direct Form-II structure:-

$$Y(z) = -\frac{13}{12} Y(z) z^{-1} - \frac{9}{24} z^{-2} Y(z) - \frac{1}{24} z^{-3} Y(z) + X(z) + 4z^{-1} X(z) + 3z^{-2} X(z)$$

$$Y(z) + \frac{13}{12} Y(z) z^{-1} + \frac{9}{24} z^{-2} Y(z) + \frac{1}{24} z^{-3} Y(z) = X(z) + 4z^{-1} X(z) + 3z^{-2} X(z)$$

$$Y(z)\left[1 + \frac{13}{12} z^{-1} + \frac{9}{24} z^{-2} + \frac{1}{24} z^{-3}\right] - X(z)\left[1 + 4z^{-1} + 3z^{-2}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 4z^{-1} + 3z^{-2}}{1 + \frac{13}{12} z^{-1} + \frac{9}{24} z^{-2} + \frac{1}{24} z^{-3}}$$

$$\frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)} = \frac{1 + 4z^{-1} + 3z^{-2}}{1 + \frac{13}{12} z^{-1} + \frac{9}{24} z^{-2} + \frac{1}{24} z^{-3}}$$

Let, $\frac{Y(z)}{W(z)} = 1 + 4z^{-1} + 3z^{-2}$

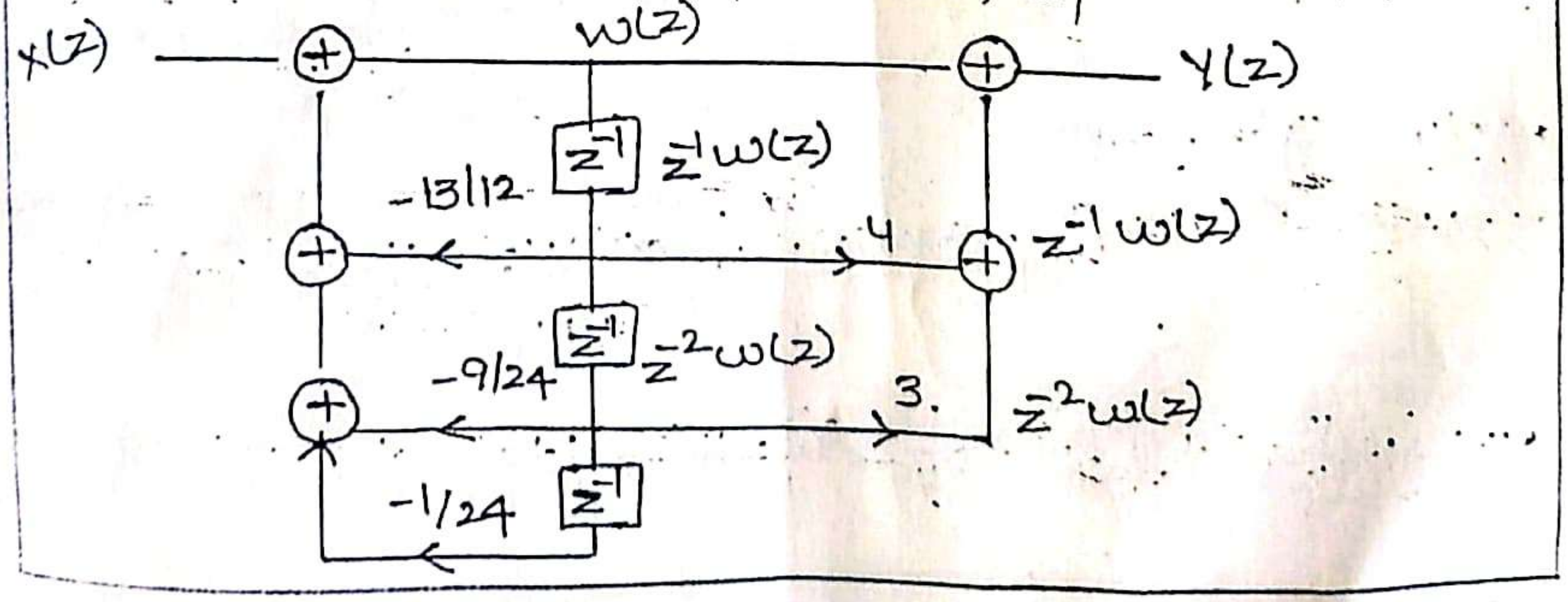
& $\frac{W(z)}{X(z)} = \frac{1}{1 + \frac{13}{12} z^{-1} + \frac{9}{24} z^{-2} + \frac{1}{24} z^{-3}}$

On cross multiplying above equations,

$$Y(z) = W(z) + 4z^{-1} W(z) + 3z^{-2} W(z)$$

$$X(z) = W(z) + \frac{13}{12} z^{-1} W(z) + \frac{9}{24} z^{-2} W(z) + \frac{1}{24} z^{3} W(z)$$

$$W(z) = X(z) - \frac{13}{12} z^{-1} W(z) - \frac{9}{24} z^{-2} W(z) - \frac{1}{24} z^{-3} W(z)$$



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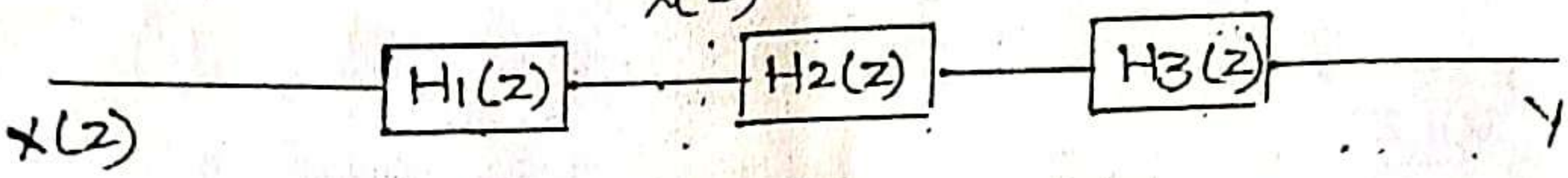
Cascade Realization:-

The cascade form structure is a series interconnected of the sub transfer functions which are realized by using the direct form structure. Hence in a cascade form realization, the given transfer function H(z) is expressed as a product of no. of subtransfer functions as shown below

$$H(z) = \frac{Y(z)}{X(z)} = k \prod_{i=1} H_i(z).$$



⇒ Obtain the cascade form realisation to the given difference equation.

$$y(n) = \frac{-13}{12} y(n-1) - \frac{9}{24} y(n-2) - \frac{1}{24} y(n-3) + x(n) + 4x(n-1) + 3x(n-2)$$

A. Given eq,

$$y(n) = -\frac{13}{12} y(n-1) - \frac{9}{24} y(n-2) - \frac{1}{24} y(n-3) + x(n) + 4x(n-1) + 3x(n-2)$$

$$y(n) + \frac{13}{12} y(n-1) + \frac{9}{24} y(n-2) + \frac{1}{24} y(n-3) = x(n) + 4x(n-1) + 3x(n-2)$$

Apply z-Transform,

$$Y(z) + \frac{13}{12} z^{-1} Y(z) + \frac{9}{24} z^{-2} Y(z) + \frac{1}{24} z^{-3} Y(z) = X(z) + 4z^{-1} X(z) + 3z^{-2} X(z)$$

$$Y(z)\left[1 + \frac{13}{12} z^{-1} + \frac{9}{24} z^{-2} + \frac{1}{24} z^{-3}\right] = X(z)\left[1 + 4z^{-1} + 3z^{-2}\right]$$

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$$\frac{Y(z)}{X(z)} = \frac{1+4z^{-1}+3z^{-2}}{1+\frac{13}{12}z^{-1}+\frac{9}{24}z^{-2}+\frac{1}{24}z^{-3}}$$

$$= \frac{(z^{-1}+1)(3z^{-1}+1)}{(1+\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})(1+\frac{1}{4}z^{-1})} = \frac{(1+z^{-1})(1+3z^{-1})}{(1+\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})(1+\frac{1}{4}z^{-1})}$$

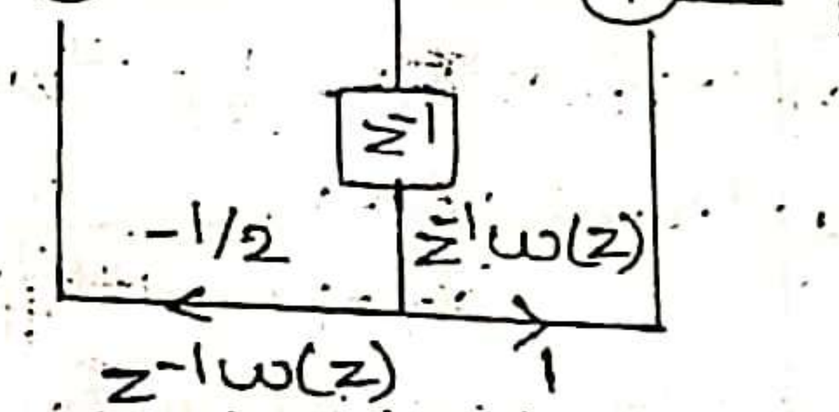
Let,

$$H(z) = H_1(z) \cdot H_2(z) \cdot H_3(z)$$

Where, $H_1(z) = \frac{(1+z^{-1})}{(1+\frac{1}{2}z^{-1})}$

$$H_2(z) = \frac{(1+3z^{-1})}{(1+\frac{1}{3}z^{-1})}$$

$$H_3(z) = \frac{1}{(1+\frac{1}{4}z^{-1})}$$



Realize $H_1(z)$, $H_2(z)$, $H_3(z)$ by using direct form-II

$$H_1(z) = \frac{Y_1(z)}{X_1(z)} = \frac{(1+z^{-1})}{(1+\frac{1}{2}z^{-1})}$$

$$H_1(z) = \frac{Y_1(z)}{W_1(z)} \cdot \frac{W_1(z)}{X_1(z)} = (1+z^{-1}) \cdot \left(\frac{1}{1+\frac{1}{2}z^{-1}}\right)$$

$$\frac{Y_1(z)}{W_1(z)} = 1+z^{-1}$$

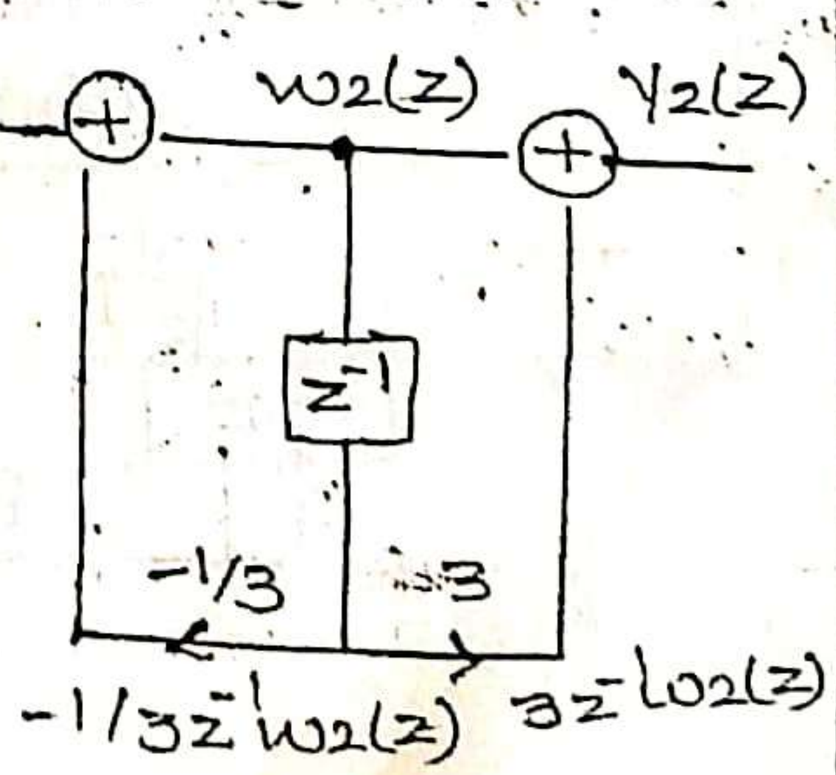
$$Y_1(z) = W_1(z) + z^{-1}W_1(z)$$

$$\frac{W_1(z)}{X_1(z)} = \frac{1}{1+\frac{1}{2}z^{-1}}$$

$$X_1(z) = W_1(z) + \frac{1}{2}z^{-1}W_1(z)$$

$$W_1(z) = X_1(z) - \frac{1}{2}z^{-1}W_1(z)$$

$$H_2(z) = \frac{Y_2(z)}{X_2(z)} = \frac{1+3z^{-1}}{1+\frac{1}{3}z^{-1}}$$



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$$H_2(z) = \frac{Y_2(z)}{W_2(z)} \cdot \frac{W_2(z)}{X_2(z)} = 1 + 3z^{-1} \cdot \frac{1}{1+\frac{1}{3}z^{-1}}$$

$$\frac{Y_2(z)}{W_2(z)} = 1 + 3z^{-1}$$

$$Y_2(z) = W_2(z) + 3z^{-1} W_2(z)$$

$$\frac{W_2(z)}{X_2(z)} = \frac{1}{1+\frac{1}{3}z^{-1}}$$

$$X_2(z) = W_2(z) + \frac{1}{3} z^{-1} \cdot W_2(z)$$

$$W_2(z) = X_2(z) - \frac{1}{3} z^{-1} W_2(z)$$

$$H_3(z) = \frac{Y_3(z)}{X_3(z)} = \frac{1}{1+\frac{1}{4}z^{-1}}$$

$$H_3(z) = \frac{Y_3(z)}{W_3(z)} \cdot \frac{W_3(z)}{X_3(z)} = 1 \cdot \frac{1}{1+\frac{1}{4}z^{-1}}$$

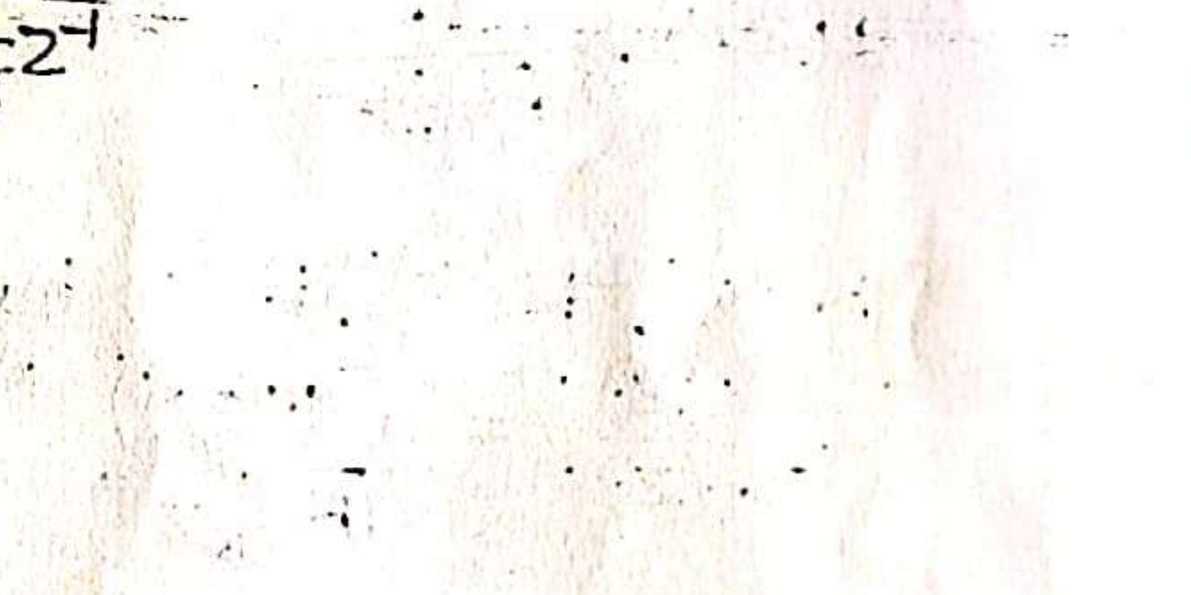
$$\frac{Y_3(z)}{W_3(z)} = 1$$

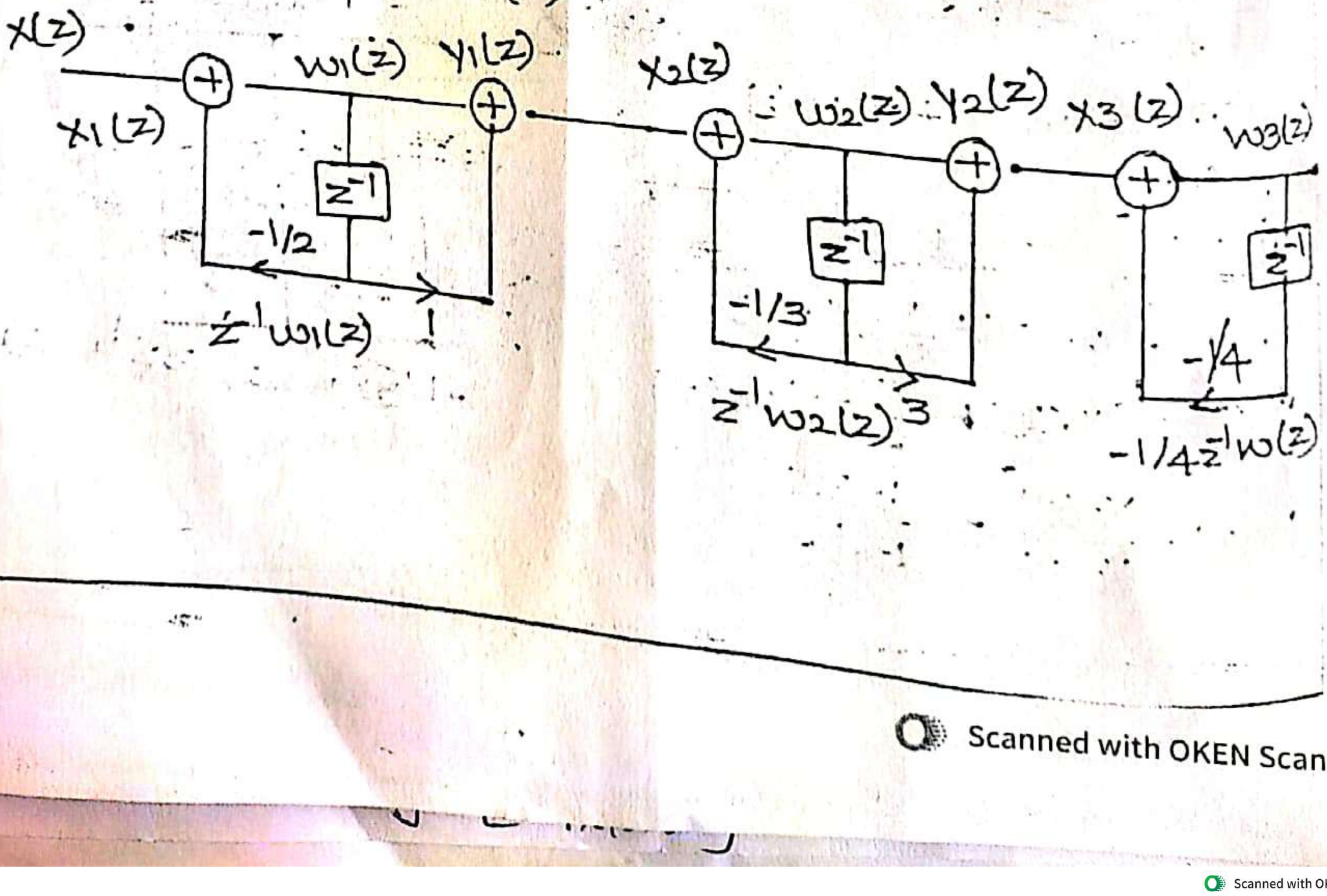
$$Y_3(z) = W_3(z)$$

$$\frac{W_3(z)}{X_3(z)} = \frac{1}{1+\frac{1}{4}z^{-1}}$$

$$X_3(z) = W_3(z) + \frac{1}{4} z^{-1} W_3(z)$$

$$W_3(z) = X_3(z) - \frac{1}{4} z^{-1} W_3(z)$$





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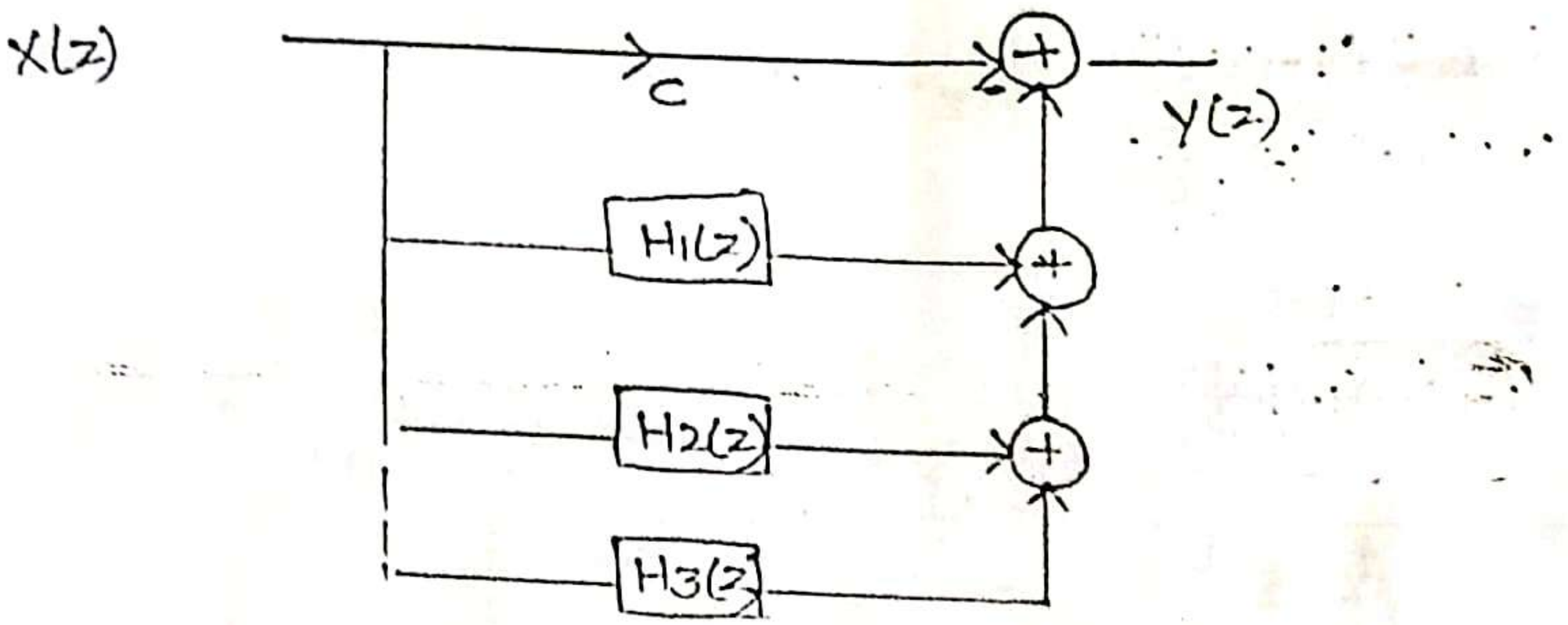
Parallel Realization:-

⇒ In parallel form realization by using partial fraction expansion that transfer function $H(z)$ is expressed as a sum of no. of subtransfer functions as shown below.

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = C + \sum_{i=1}^{K} H_i(z)$$



⇒ Obtain the parallel form realisation to the given difference equation.

$$y(n) = \frac{-13}{12} y(n-1) - \frac{9}{24} y(n-2) - \frac{1}{24} y(n-3) + x(n) + 4x(n-1) + 3x(n-2)$$

Given,

$$y(n) = \frac{-13}{12} y(n-1) - \frac{9}{24} y(n-2) - \frac{1}{24} y(n-3) + x(n) + 4x(n-1) + 3x(n-2)$$

Apply z-Transform.

$$y(z) = \frac{-13}{12} z^{-1} y(z) - \frac{9}{24} z^{-2} y(z) - \frac{1}{24} z^{-3} y(z) + x(z) + 4z^{-1} x(z) + 3z^{-2} x(z)$$

$$y(z)\left[1 + \frac{13}{12} z^{-1} + \frac{9}{24} z^{-2} + \frac{1}{24} z^{-3}\right] = x(z)\left[1 + 4z^{-1} + 3z^{-2}\right]$$

$$H(z) = \frac{y(z)}{x(z)} = \left[\frac{1 + 4z^{-1} + 3z^{-2}}{1 + \frac{13}{12} z^{-1} + \frac{9}{24} z^{-2} + \frac{1}{24} z^{-3}}\right]$$

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$$\frac{y(z)}{x(z)} = \frac{(1+z^{-1})(1+3z^{-1})}{(1+\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})(1+\frac{1}{4}z^{-1})}$$

Apply partial fractions expansion,

$$\frac{(1+z^{-1})(1+3z^{-1})}{(1+\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})(1+\frac{1}{4}z^{-1})} = \frac{A}{(1+\frac{1}{2}z^{-1})} + \frac{B}{(1+\frac{1}{3}z^{-1})} + \frac{C}{(1+\frac{1}{4}z^{-1})}$$

$$A = \left.\frac{(1+z^{-1})(1+3z^{-1})}{(1+\frac{1}{3}z^{-1})(1+\frac{1}{4}z^{-1})}\right| z^{-1} = -2$$

$$= \frac{(-1)(-5)}{(1-\frac{2}{3})(1-\frac{1}{2})}$$

$$= \frac{5}{1-\frac{2}{3}-\frac{1}{2}+\frac{1}{3}} = \frac{5}{1/6} = 30$$

$$= \frac{5}{6\frac{-4-3+2}{6}}$$

$$\boxed{A = 30}$$

$$B = \left.\frac{(1+z^{-1})(1+3z^{-1})}{(1+\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})}\right| z^{-1} = -3$$

$$= \frac{(-2)(-8)}{(1-\frac{3}{2})(1-\frac{3}{4})} = \frac{16}{1-\frac{3}{4}-\frac{3}{2}+\frac{9}{8}} = \frac{16}{-1/8} = -128$$

$$\therefore \boxed{B = -128}$$

$$C = \left.\frac{(1+z^{-1})(1+3z^{-1})}{(1+\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})}\right| z^{-1} = -4$$

$$= \frac{(-3)(-11)}{(1-2)(1-\frac{4}{3})} = \frac{33}{-1(-\frac{1}{3})} = \frac{33}{1/3} = 99$$

$$\therefore \boxed{C = 99}$$

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$$\frac{Y(z)}{X(z)} = \frac{30}{\left(1+\frac{1}{2}z^{-1}\right)} - \frac{128}{\left(1+\frac{1}{3}z^{-1}\right)} + \frac{99}{\left(1+\frac{1}{4}z^{-1}\right)}$$

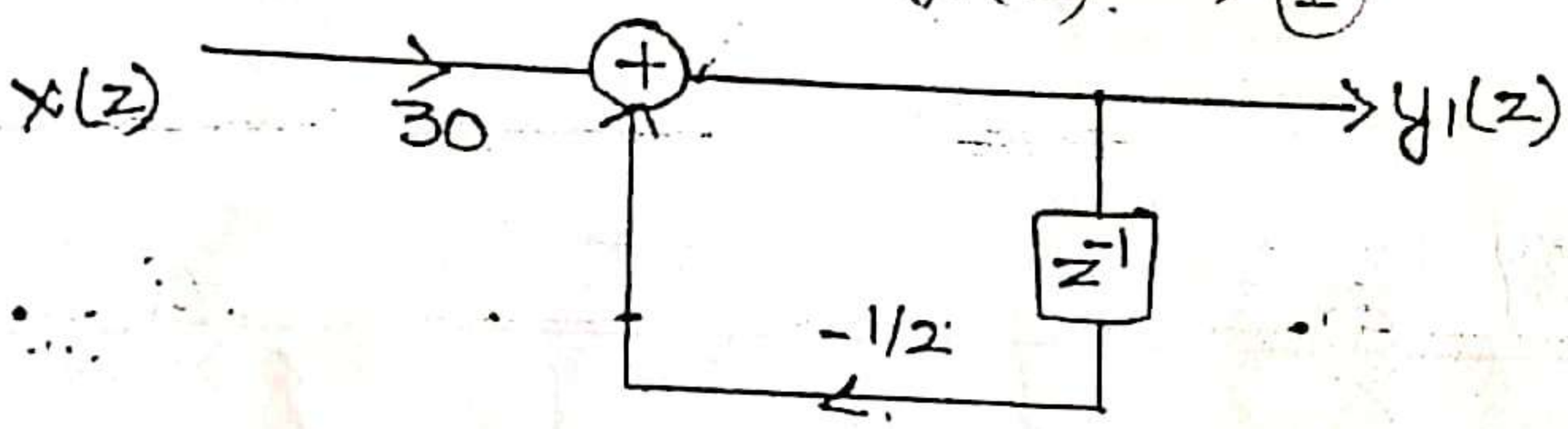
$$Y(z) = \frac{30X(z)}{\left(1+\frac{1}{2}z^{-1}\right)} - \frac{128X(z)}{\left(1+\frac{1}{3}z^{-1}\right)} + \frac{99X(z)}{\left(1+\frac{1}{4}z^{-1}\right)}$$

$$Y(z) = Y_1(z) + Y_2(z) + Y_3(z)$$

$$Y_1(z) = \frac{30X(z)}{\left(1+\frac{1}{2}z^{-1}\right)}$$

$$y_1(z) + \frac{1}{2}z^{-1}y_1(z) = 30X(z)$$

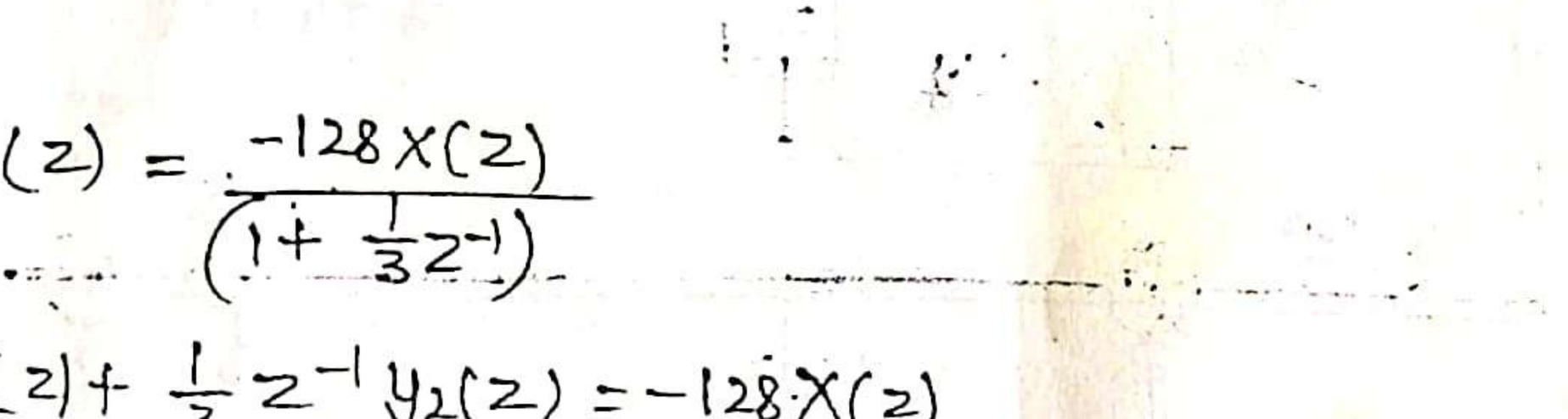
$$y_1(z) = 30X(z) - \frac{1}{2}z^{-1}y_1(z) \longrightarrow (1)$$



$$y_2(z) = \frac{-128X(z)}{\left(1+\frac{1}{3}z^{-1}\right)}$$

$$y_2(z) + \frac{1}{3}z^{-1}y_2(z) = -128X(z)$$

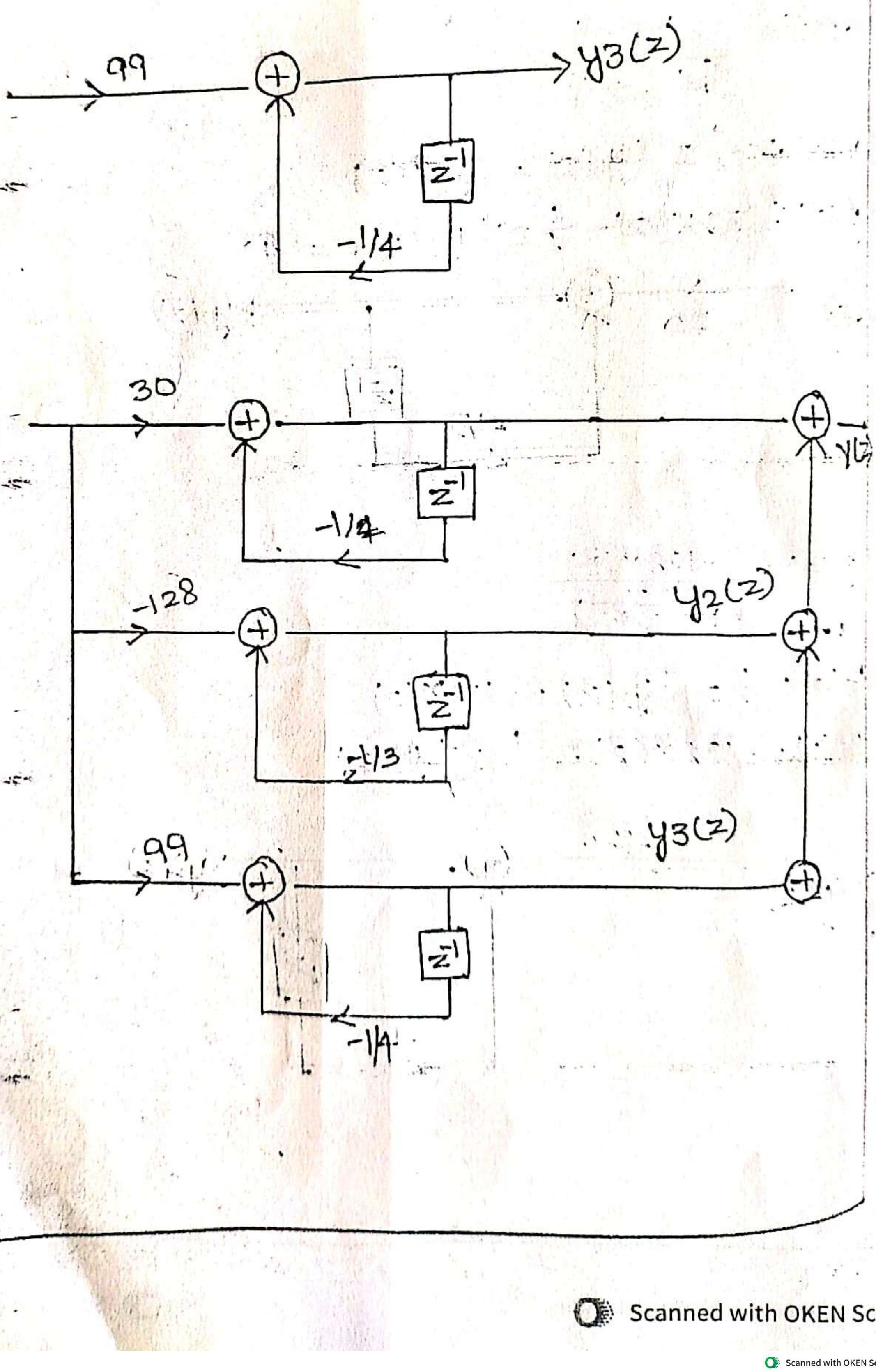
$$y_2(z) = -128X(z) - \frac{1}{3}z^{-1}y_2(z)$$



$$Y_3(z) = \frac{99X(z)}{\left(1+\frac{1}{4}z^{-1}\right)}$$

$$Y_3(z) + \frac{1}{4}z^{-1}Y_3(z) = 99X(z)$$

$$Y_3(z) = 99X(z) - \frac{1}{4}z^{-1}Y_3(z)$$



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Comparision between IIR and FIR filters.

IIR	FIR
1. IIR has infinite no. of samples of impulse response.	1. FIR has finite no. of samples of impulse response.
2. The o/p depends upon present i/p, past i/p & past o/p samples.	2. The o/p depends upon present i/p & past i/p samples.
3. Both poles & zeroes are present.	3. Only zeroes are present.
4. It is recursive s/m.	4. It is non-recursive s/m.
5. IIR filters is a closed loop system.	5. It is a open loop system.
6. More complex to implement.	6. Less complex to implement.
7. Realibility is not always gauranteed.	7. Realibility is always gauranteed.
8. It does not have linear phase characteristics.	8. It has linear phase characteristics.

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UNIT - IV

FINITE IMPULSE RESPONSE FILTERS

An FIR filter is a filter whose impulse response is finite. The design of FIR filter consists of all the finite no. of samples of impulse response.

* The response of FIR filter depends on present and past inputs but not on past outputs.

* The general form of difference equation of an FIR filter is described by

$$y(n) = \sum_{k=0}^{M} b_k x(n-k)$$

$$Y(k) = \sum_{k=0}^{M} b_k \cdot z^{-k} \cdot x(z)$$

$$\frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{}$$

$$\therefore H(z) = \sum_{k=0}^{M} b_k z^{-k}$$

Fourier Series method of designing FIR filter

Either Fourier Series (or) Windowing Technique.

$H_d(e^{j\omega})$ → Frequency Response (Given)

$H(z)$ → Transfer Function (Findout)

$H(z)$ → Practical Realization (Implementation) is $H(z)$

step 1:- $H_d(n)$ → for windowing Technique

step 2:- $h(n)$ → {$h(n) = H_d(n)$ * Particular window}

Step 3:- $H(z)$ →

Step 4:- $H'(z)$

Step 5:- Frequency Response (If is given)

Formulas for filter design:-

① $H_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega$.

② $n=0$ apply L-Hospital rule where n is $\left(\frac{N-1}{2}\right)$

③ Theoritical Transfer Function.

$$H(z) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} \left[h(n) (z^n + z^{-n}) \right].$$

④ Practical Transfer function

$$H'(z) = z^{-1}\left(\frac{N-1}{2}\right) H(z).$$

⑤ $\bar{H}(e^{j\omega}) = \sum_{n=0}^{\frac{N-1}{2}} a(n) \cos\omega(n)$

where $a(0) = h\left(\frac{N-1}{2}\right)$

Design of FIR filters using Fourier Series Meth.

The desired frequency response of FIR filter can be represented by the Fourier Series

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d(n) e^{-j\omega n}$$

$h_d(n)$ is the impulse sequence of the given filter it's having.

Design Steps:-

(1) For the desired frequency response $H_d(e^{j\omega})$ find the desired impulse sequence of the filter

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} \, d\omega$$

The impulse response $h_d(n)$ is having infinite duration, But we are design FIR filter which should not finite duration. The filter coefficients from $-\infty \le n \le \infty$.

(2) Truncate $h_d(n)$ at $n = \pm\left(\frac{N-1}{2}\right)$ to get the finite duration sequence $h(n)$.

$$h(n) = \begin{cases} h_d(n) ; |n| \le \frac{N-1}{2} \\ 0 ; \text{otherwise.} \end{cases}$$

For a symmetrical impulse response symmetric at $n=0$ $\therefore h(n) = h(-n)$

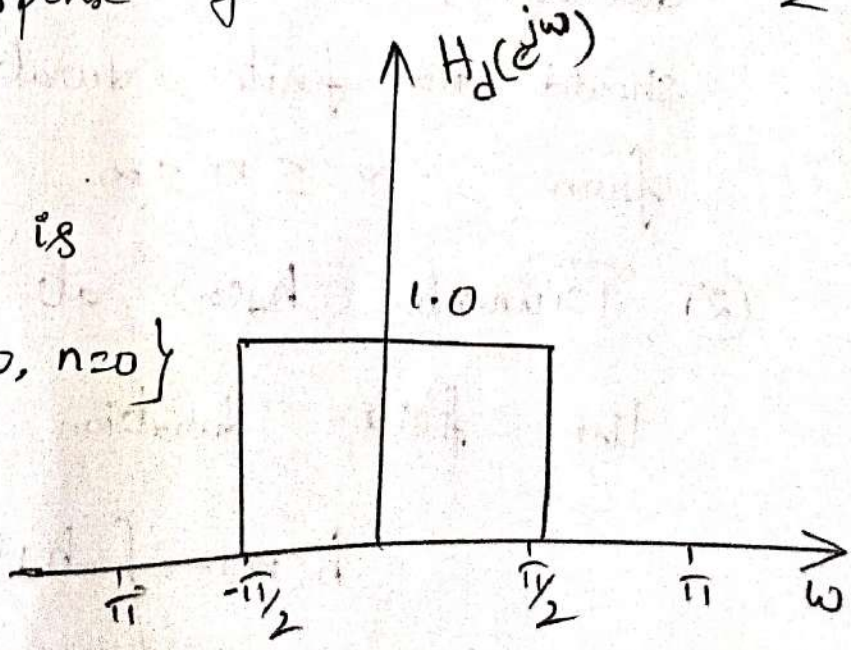
(3) The transfer function $H(Z)$ of the realizable filter is.

$$H(Z) = Z^{-(N-1/2)} \left[h(0) + \sum_{n=1}^{N-1} h(n) \left[Z^{n} + Z^{-n} \right] \right]$$

Problems:

(1) Design an ideal Low Pass Filter with a frequency response $H_d(e^{j\omega}) = \begin{cases} 1 & ; -\pi/2 \le \omega \le \pi/2 \\ 0 & ; \text{otherwise} \end{cases}$, find the values of $h(n)$ for $N=11$, and find $H(Z)$. plot the magnitude response.

Sol:- The frequency response of LPF with $\omega_c = \pi/2$ as shown below.



The given filter is symmetric about $\{\alpha=0, n=0\}$

$\left[\because \alpha = \frac{N-1}{2} = \frac{11-1}{2} = 5 \right]$

Step ① Find the desired impulse sequence of the filter.

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega.$$

$$= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} (1)\cdot e^{j\omega n}\, d\omega \longrightarrow ①$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{2\pi jn}\left[e^{j\pi/2 n} - e^{-j\pi/2 n}\right]$$

$$= \frac{1}{\pi n}\cdot\left[\frac{e^{j\pi/2 n} - e^{-j\pi/2 n}}{2j}\right] \quad \left\{\because \frac{e^{j\theta} - e^{-j\theta}}{2j}\right.$$

$$\therefore h_d(n) = \frac{1}{\pi n}\cdot \sin\left(\frac{n\pi}{2}\right)$$

$$\therefore h_d(n) = \frac{\sin \pi/2\, n}{\pi n} \qquad -\infty \le n \le \infty$$

Step ②: Truncating the $h_d(n)$ to N=11 samples

$$h(n) = \begin{cases} \frac{1}{\pi n}\sin\left(\frac{n\pi}{2}\right) & ; |n| \le 5 \\ 0 & \text{otherwise.} \end{cases}$$

For n=0, h(n) becomes indeterminate, so put n=0 in the below eqn.

$$h(0) = \lim_{n\to 0} \frac{\sin \frac{n\pi}{2}}{n\pi} = \frac{1}{2\pi}\lim_{n\to 0} \frac{\sin \frac{\pi n}{2}}{\frac{n\pi}{2}}$$

$$\therefore \lim \sin\theta$$

(8)

$$h(0) = h_d(0) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{0} \cdot d\omega$$

$$h(0) = \frac{1}{2\pi} [\omega]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{2\pi} [\pi/2 + \pi/2] \Rightarrow \frac{1}{2\pi} \left[\frac{2\pi}{2}\right]$$

$$\therefore h(0) = 1/2 = 0.5$$

for $n=1$

$$h(1) = h(-1) = \frac{\sin \frac{n\pi}{2}}{n\pi} \Rightarrow \frac{\sin(1)\pi/2}{1\cdot\pi} = \frac{1}{\pi}$$

$$= 0.3183$$

for $n=2$

$$h(2) = h(-2) = \frac{\sin 2\pi/2}{2\pi} = \frac{\sin \pi}{2\pi} = 0$$

for $n=3$

$$h(3) = h(-3) = \frac{\sin 3\pi/2}{3\pi} = -\frac{1}{3\pi} = -0.106$$

for $n=4$

$$h(4) = h(-4) = \frac{\sin 4\pi/2}{4\pi} = \frac{\sin 2\pi}{4\cdot\pi} = 0$$

for $n=5$

$$h(5) = h(-5) = \frac{\sin \frac{5\pi}{2}}{5\pi} = 1/5\pi = 0.0636.$$

$h(0) = 1/2$, $h(1) = h(-1) = 0.3183$, $h(2) = h(-2) = 0$,

$h(3) = h(-3) = -0.106$, $h(4) = h(-4) = 0$, $h(5) = h(-5) = 0.06366$.

Step (8) Find the transfer function H(z)

$$H(z) = z^{-1(N-1/2)}\left[h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)\,[z^n + z^{-n}]\right]$$

$$H(z) = z^{-1(11-1/2)}\left[\frac{1}{2} + \sum_{n=1}^{5} h(n)\,[z^n + z^{-n}]\right]$$

$$H(z) = z^{-5}\{0.5 + h(1)[z^1 + z^{-1}] + h(2)[z^{+2} + z^{-2}] + h(3)[z^3 + z^{-3}] + h(4)[z^4 + z^{-4}] + h(5)[z^5 + z^{-5}]\}$$

$$\Rightarrow z^{-5}\{0.5 + 0.3183(z^1 + z^{-1}) + 0 + (-0.106(z^3 + z^{-3}) + 0 + 0.06366(z^5 + z^{-5})\}$$

$$\Rightarrow z^{-5}\{0.5 + 0.3183z^1 + 0.3183z^{-1} - 0.106z^3 - 0.1062^{-3} + 0.06366z^5 + 0.06366z^{-5}\}$$

$$\therefore H(z) = 0.5z^{-5} + 0.3183z^{-4} + 0.3183z^{-6} - 0.106z^{-2} - 0.106z^{-8} + 0.06366z^0 + 0.06366z^{-10}.$$

∴ The transfer function of FIR filter is

$$H(z) = 0.06366 - 0.106z^{-2} + 0.3183z^{-4} + 0.5z^{-5} + 0.3183z^{-6} - 0.106z^{-8} + 0.06366z^{-10}$$

Filter coefficients

$$h(n) = \{0.06366,\ 0,\ -0.106,\ 0,\ 0.3183,\ \underset{\uparrow}{0.5},\ 0.3183,\ 0,\ -0.106,\ 0,\ 0.06366\}$$

From the above the filter coefficients of causal filter are given by

$h(0) = h(10) = 0.06366$, $h(1) = h(9) = 0$, $h(2) = h(8) = -0.1006$

$h(3) = h(7) = 0$, $h(4) = h(6) = 0.3183$, $h(5) = 0.5$.

The frequency response is given by

$$\bar{H}(e^{j\omega}) = \sum_{n=0}^{5} a(n) \cos \omega n \quad \text{where.}$$

$$a(0) = h\left(\frac{N-1}{2}\right) = h(5) = 0.5$$

$$a(n) = 2h\left[\frac{N-1}{2} - n\right]$$

$$a(1) = 2h(5-1) = 2h(4) = 0.6366$$

$$a(2) = 2h(5-2) = 2h(3) = 0$$

$$a(3) = 2h(5-3) = 2h(2) = -0.212$$

$$a(4) = 2h(5-4) = 2h(1) = 0$$

$$a(5) = 2h(5-5) = 2h(0) = 0.127.$$

$$\bar{H}(e^{j\omega}) = 0.5 + 0.6366 \cos\omega - 0.212 \cos 3\omega + 0.127 \cos 5\omega$$

The magnitude in dB is calculated by varying ω from 0 to π and tabulated below:

The magnitude $|H(e^{j\omega})|_{dB} = 20 \log |\bar{H}(e^{j\omega})|$.

ω(in degree)	0	10	20	30	40	50	60	70	80
$\lvert H(e^{j\omega})\rvert_{dB}$	0.4	0.21	-0.26	-0.517	-0.21	0.42	0.77	0.21	-1.79

90	100	110	120	130	140	150	160	170	180
-6	-14.56	-31.89	-20.6	-26	-32	-24.7	-30.55	-32	-26

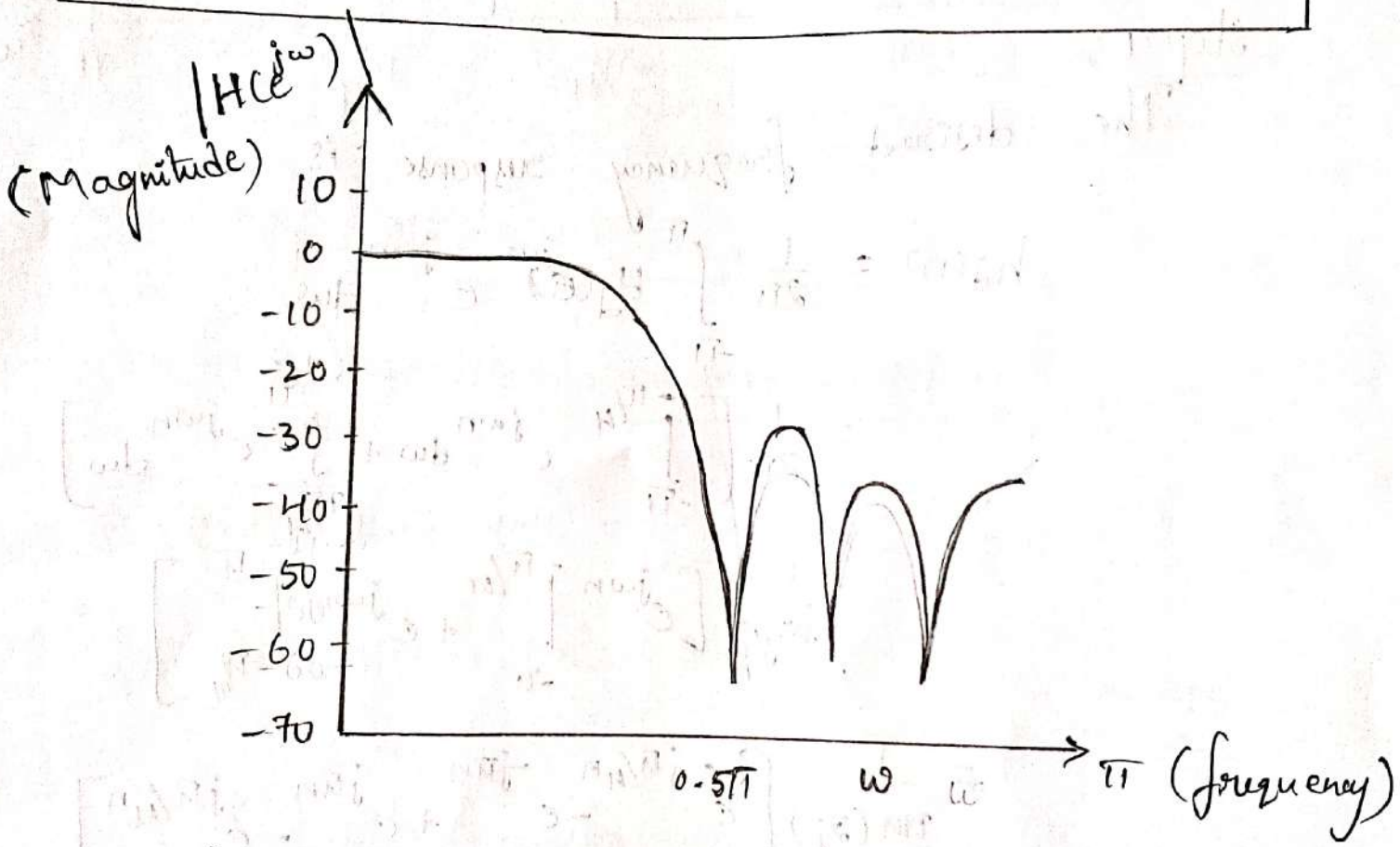


fig: Frequency Response of Low Pass filter.

(2) Design an ideal High Pass filter with a frequency response $H_d(e^{j\omega}) = 1$ for $\pi/4 \le \omega \le \pi$
$= 0$ for $|\omega| \le \pi/4$. Find the values of $h(n)$ for $N = 11$. Find $H(z)$. Plot the magnitude response.

Sol:-

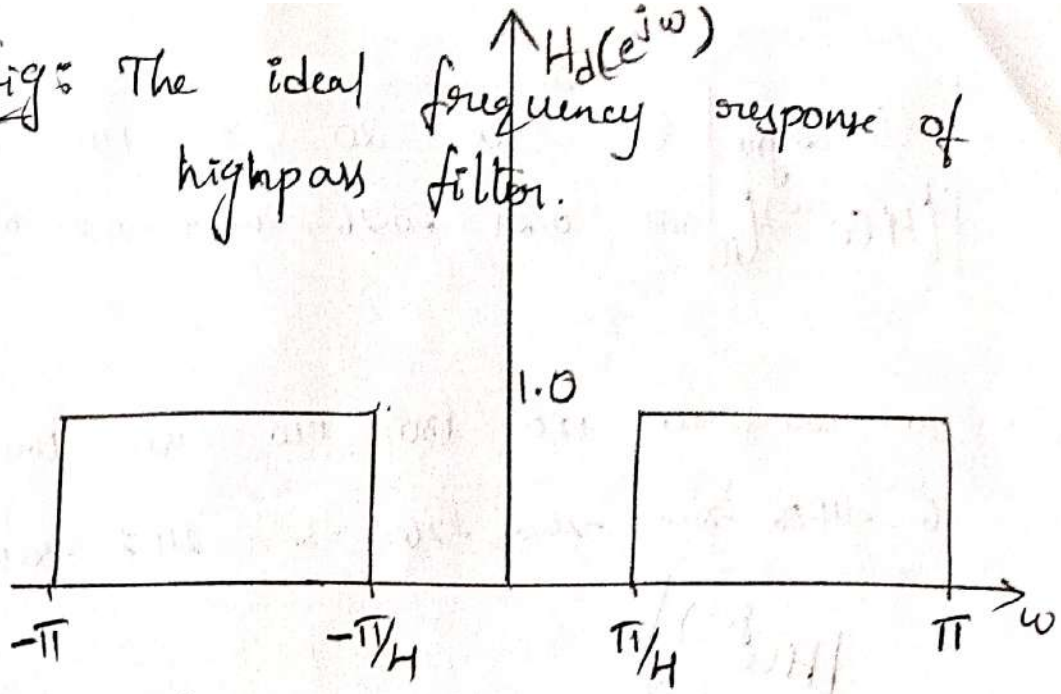


fig: The ideal frequency response of highpass filter.

Step 1:

The desired frequency response is

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega.$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{-\pi/4} e^{j\omega n}\, d\omega + \int_{\pi/4}^{\pi} e^{j\omega n}\, d\omega\right]$$

$$= \frac{1}{2\pi jn}\left[e^{j\omega n}\Big|_{-\pi}^{-\pi/4} + e^{j\omega n}\Big|_{\pi/4}^{\pi}\right]$$

$$= \frac{1}{\pi n(2j)}\left[e^{-j\pi/4 n} - e^{-j\pi n} + e^{j\pi n} - e^{j\pi/4 n}\right]$$

$$= \frac{1}{\pi n}\left[\sin \pi n - \sin \frac{\pi}{4} n\right] \qquad -\infty \le n \le \infty$$

$$\therefore h_d(n) = \frac{\sin \pi n - \sin \pi/4\, n}{\pi n}$$

Step 2: Truncating $h_d(n)$ to 11 samples, we have

$$h(n) = h_d(n) \quad \text{for } |n| \le 5$$

$$0 \quad \text{otherwise.}$$

For $n=0$; $h(0) = \lim_{n\to 0} \frac{\sin \pi n}{\pi n} - \lim_{n\to 0} \frac{\sin \pi/4 n}{\pi n}$

$h(0) = \left[1 - 1/4\right]$

$h(0) = 3/4 = 0.75$

$\left\{ \because \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 \right.$

$\left. \lim_{\theta\to 0} \frac{\sin n\theta}{\theta} = n \right\}$

for $n=1$; $h(1) = h(-1) = \frac{\sin n\pi - \sin \pi/4 n}{\pi n}$

$= \frac{\sin \pi - \sin \pi/4}{\pi} = -0.225$

for $n=2$; $h(2) = h(-2) = \frac{\sin 2\pi - \sin \pi/2}{2\pi} = -0.159$

for $n=3$; $h(3) = h(-3) = \frac{\sin 3\pi - \sin \frac{3\pi}{4}}{3\pi} = -0.075$

for $n=4$; $h(4) = h(-4) = \frac{\sin 4\pi - \sin \pi}{4\pi} = 0$

for $n=5$; $h(5) = h(-5) = \frac{\sin 5\pi - \sin \frac{5\pi}{4}}{5\pi} = 0.045.$

Step 3:- The transfer function of filter is given by

$$H(z) = h(0) + \sum_{n=1}^{(N-1/2)} \left[h(n) (z^n + z^{-n}) \right]$$

$$= 0.75 + \sum_{n=1}^{5} \left[h(n) (z^n + z^{-n}) \right]$$

$= 0.75 - 0.$ (illegible)

$$H'(z) = z^{-1} \left[h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n) (z^n + z^{-n}) \right]$$

$$H(z) = z^{-5}\{0.75 - 0.225(z+z^{-1}) - 0.159(z^2+z^{-2})$$

$$-0.075(z^3+z^{-3}) + 0 + 0.045(z^5+z^{-5})\}$$

$$= z^{-5}\{0.075z^5 - 0.225z + 0.225z^{-1} - 0.159z^2 - 0.159z^{-2}$$

$$-0.075z^3 - 0.075z^{-3} + 0.045z^5 + 0.045z^{-5}\}$$

$$= 0.75z^{-5} - 0.225z^{-4} - 0.225z^{-6} - 0.159z^{-3} - 0.159z^{-7} -$$

$$0.075z^{-2} - 0.075z^{-8} + 0.045z^{0} + 0.045z^{-10}$$

$$H(z) = 0.045 - 0.075z^{-2} - 0.159z^{-3} - 0.225z^{-4} + 0.75z^{-5} -$$

$$0.225z^{-6} - 0.159z^{-7} - 0.075z^{-8} + 0.045z^{-10}$$

The filter coefficients of causal filter are

$h(0) = h(10) = 0.045$, $h(1) = h(9) = 0$, $h(2) = h(8) = -0.075$

$h(3) = h(7) = -0.159$, $h(4) = h(6) = -0.225$, $h(5) = 0.75$

$$H(e^{j\omega}) = \sum_{n=0}^{N-1/2} a(n)\cos\omega n. \quad \text{where}$$

$$a(0) = h\left(\frac{N-1}{2}\right) = h(5) = 0.75$$

$$a(n) = 2h\left(\frac{N-1}{2} - n\right)$$

$$a(1) = 2h(5-1) = 2h(4) = -0.45$$

$$a(2) = 2h(5-2) = 2h(3) = -0.318$$

$$a(3) = 2h(5-3) = 2h(2) = -0.15$$

$a(4) = 2h(5-4) = 2h(1) = 0$

$a(5) = 2h(5-5) = 2h(0) = 0.09.$

$\overline{H}(e^{j\omega}) = a(0) + a(1)\cos\omega + a(2)\cos 2\omega + a(3)\cos 3\omega$

$+ a(4)\cos 4\omega + a(5)\cos 5\omega.$

$= 0.75 - 0.45\cos\omega - 0.318\cos 2\omega - 0.15\cos 3\omega + 0.09\cos 5\omega.$

ω deg.	0	10	20	30	40	50
$\overline{H}(e^{j\omega})$	-0.08	-0.66	-0.0086	0.122	0.34	0.61
$\lvert H(e^{j\omega})\rvert$ dB	-22	-23.62	-41.3	-18.2	-9.36	-4.2

60	70	80	90	100	110	120	130
0.88	1.05	1.11	1.07	0.98	0.93	0.94	0.998
-1.1	0.504	0.95	0.58	-0.132	-0.625	-0.537	-0.637

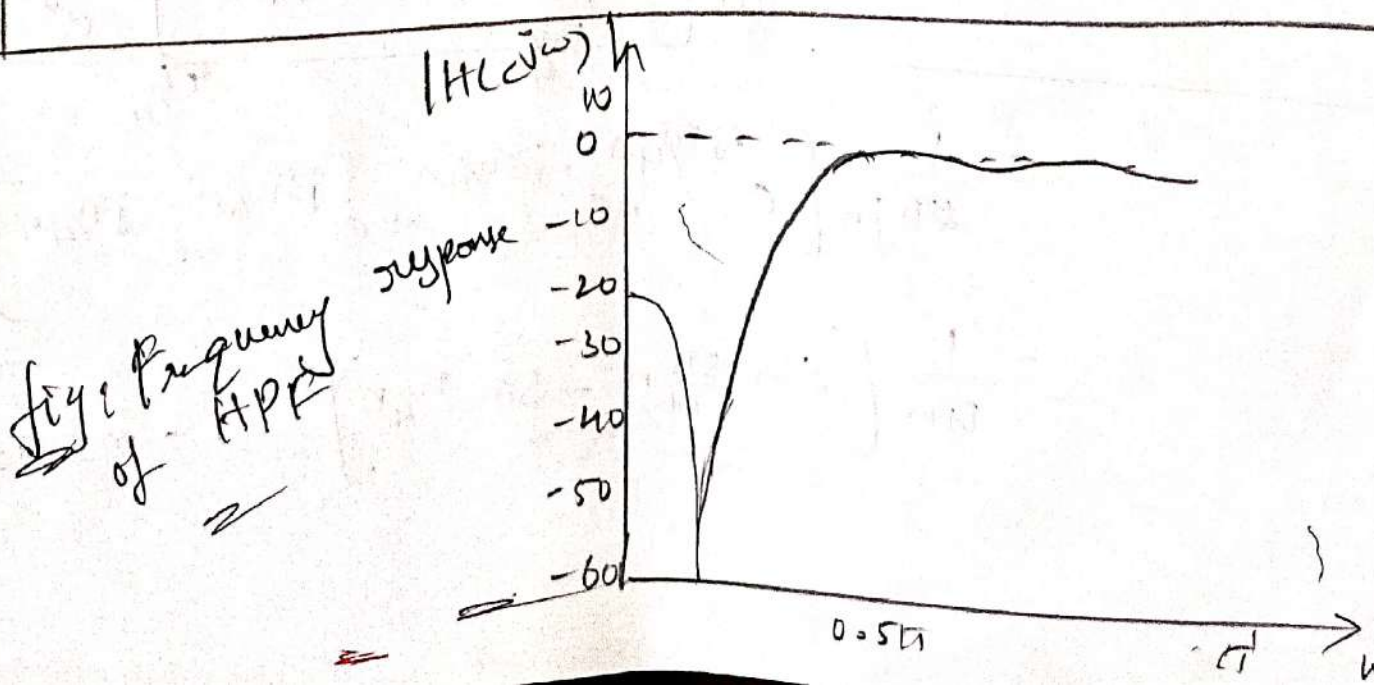


Fig: Frequency response of HPF

Example 6.7 Design an ideal bandpass filter with a frequency response

$$H_d(e^{j\omega}) = 1 \quad \text{for} \quad \frac{\pi}{4} \le |\omega| \le \frac{3\pi}{4}$$
$$= 0 \quad \text{otherwise}$$

Find the values of $h(n)$ for $N = 11$ and plot the frequency response.

Solution

The ideal frequency response of the filter shown in Fig. 6.12. We know

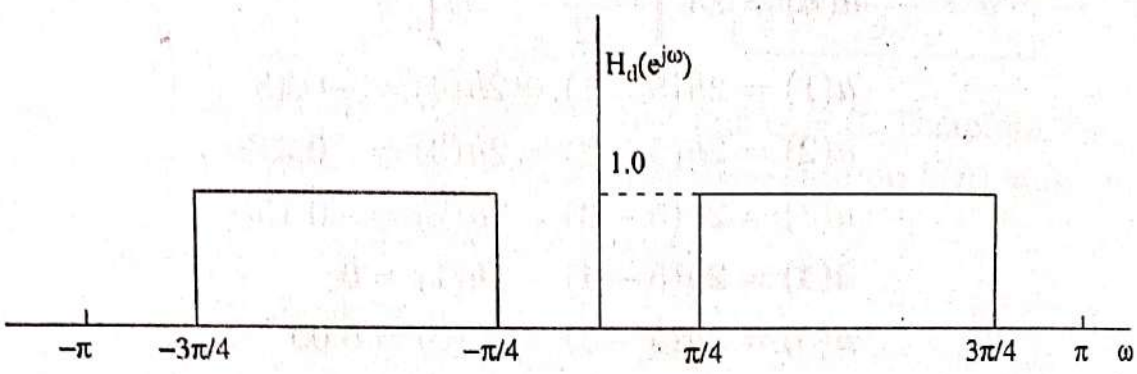


Fig. 6.12 Ideal frequency response of Bandpass filter of example 6.7.

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}$$
$$= \frac{1}{2\pi}\left[\int_{\frac{-3\pi}{4}}^{\frac{-\pi}{4}} e^{j\omega n}d\omega + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} e^{j\omega n}d\omega\right]$$
$$= \frac{1}{2\pi jn}\left[e^{-j\pi n/4} - e^{-j3\pi n/4} + e^{j3\pi n/4} - e^{j\pi n/4}\right]$$
$$= \frac{1}{\pi n}\left[\sin\frac{3\pi}{4}n - \sin\frac{\pi}{4}n\right] \qquad -\infty \le n \le \infty \tag{6.63}$$

Truncating $h_d(n)$ to 11 samples, we have

$$h(n) = h_d(n) \quad \text{for} \quad |n| \le 5$$
$$= 0 \quad \text{otherwise}$$

The filter coefficients are symmetrical about $n = 0$ satisfying the condition $h(n) = h(-n)$.

For $n = 0$

$$h(0) = \frac{1}{2\pi}\left[\int_{\frac{-3\pi}{4}}^{\frac{-\pi}{4}} d\omega + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} d\omega\right]$$
$$= \frac{1}{2\pi}\left[-\frac{\pi}{4} + \frac{3\pi}{4} + \frac{3\pi}{4} - \frac{\pi}{4}\right] = \frac{1}{2} = 0.5$$

$$h(1) = h(-1) = \frac{\sin\frac{3\pi}{4} - \sin\frac{\pi}{4}}{\pi} = 0$$

$$h(2) = h(-2) = \frac{\sin\frac{3\pi}{2} - \sin\frac{\pi}{2}}{2\pi} = \frac{-2}{2\pi} = -0.3183$$

$$h(3) = h(-3) = \frac{\sin\frac{9\pi}{4} - \sin\frac{3\pi}{4}}{3\pi} = 0$$

$$h(4) = h(-4) = \frac{\sin 3\pi - \sin\pi}{4\pi} = 0$$

$$h(5) = h(-5) = \frac{\sin\frac{15\pi}{4} - \sin\frac{5\pi}{4}}{5\pi} = 0$$

The transfer function of the filter is

$$\begin{aligned} H(z) &= h(0) + \sum_{n=1}^{\frac{N-1}{2}} \left[h(n)\left(z^n + z^{-n}\right)\right] \\ &= 0.5 - 0.3183(z^2 + z^{-2}) \end{aligned} \tag{6.64}$$

The transfer function of the realizable filter is

$$\begin{aligned} H'(z) &= z^{-5}\left[0.5 - 0.3183(z^2 + z^{-2})\right] \\ &= -0.3183z^{-3} + 0.5z^{-5} - 0.3183z^{-7} \end{aligned} \tag{6.65}$$

The filter coefficients of the causal filters are

$$h(0) = h(10) = h(1) = h(9) = h(2) = h(8) = h(4) = h(6) = 0$$

$$h(3) = h(7) = -0.3183$$

$$h(5) = 0.5$$

$$\overline{H}(e^{j\omega}) = \sum_{n=1}^{\frac{N-1}{2}} a(n)\cos\omega n$$

$$a(0) = h\left(\frac{N-1}{2}\right) = h(5) = 0.5$$

$$a(n) = 2h\left[\frac{N-1}{2} - n\right]$$

$$a(1) = 2h(5-1) = 2h(4) = 0$$

$$a(2) = 2h(5-2) = 2h(3) = -0.6366$$

$$a(3) = 2h(5-3) = 2h(2) = 0$$

$$a(4) = 2h(5-4) = 2h(1) = 0$$

$$a(5) = 2h(5-5) = 2h(0) = 0$$

$$\overline{H}(e^{j\omega}) = 0.5 - 0.6366\cos 2\omega \tag{6.66}$$

ω (in degrees)	0	20	30	45	60	75	90
$\overline{H}(e^{j\omega})$	−0.1366	0.012	0.1817	0.5	0.818	1.05	1.1366
$\lvert H(e^{j\omega})\rvert_{dB}$	−17.3	−38.17	−14.8	−6.02	−1.74	0.4346	1.11

105	120	135	150	160	180
1.05	0.818	0.5	0.1817	0.012	−0.1366
0.4346	−1.74	−6.02	−14.8	−38.17	−17.3

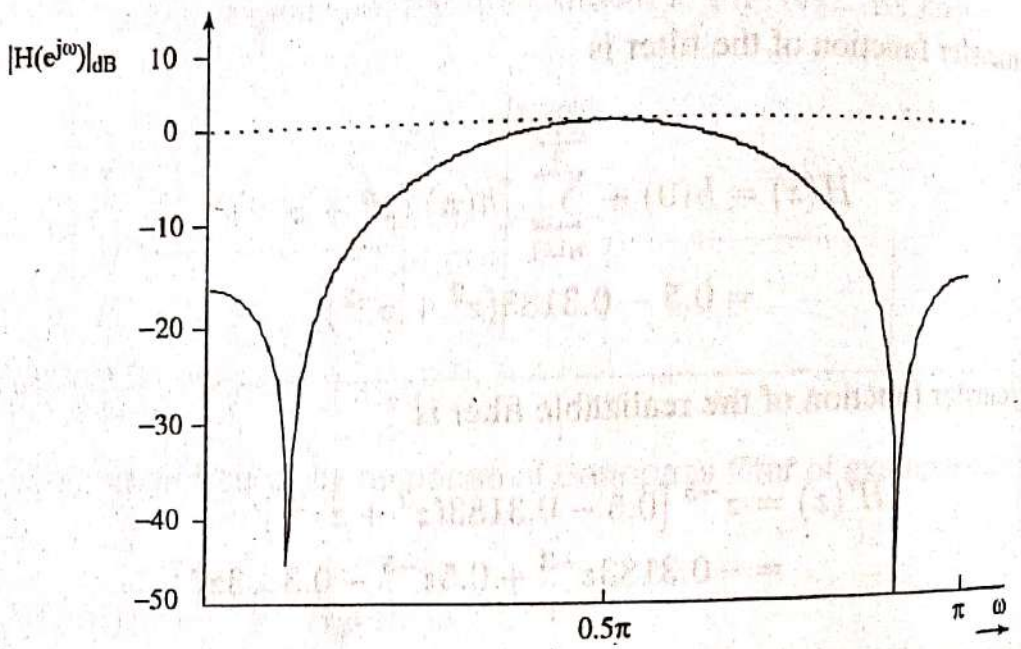


Fig. 6.13 Frequency response of Bandpass filter of example 6.7.

Example 6.8 Design an ideal bandreject filter with a desired frequency response

$$H_d(e^{j\omega}) = 1 \quad \text{for} \quad |\omega| \leq \frac{\pi}{3} \quad \text{and} \quad |\omega| \geq \frac{2\pi}{3}$$
$$= 0 \quad \text{otherwise}$$

Find the value of $h(n)$ for $N = 11$. Find $H(z)$. Plot the magnitude response.

Solution

The desired frequency response is shown in Fig. 6.14.

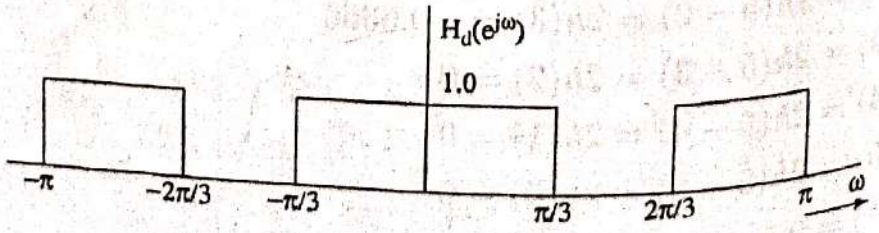


Fig. 6.14 Frequency response of Bandreject filter of example 6.8.

We know

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{-2\pi/3} e^{j\omega n}d\omega + \int_{-\pi/3}^{\pi/3} e^{j\omega n}d\omega + \int_{2\pi/3}^{\pi} e^{j\omega n}d\omega\right]$$

$$= \frac{1}{2\pi jn}\left[e^{-j2\pi n/3} - e^{-j\pi n} + e^{j\pi n/3} - e^{-j\pi n/3} + e^{j\pi n} - e^{j2\pi n/3}\right]$$

$$= \frac{1}{\pi n}\left[\sin \pi n + \sin\frac{\pi}{3}n - \sin\frac{2\pi}{3}n\right] \qquad -\infty \le n \le \infty$$

Truncating $h_d(n)$ to 11 samples, we have

$$h(n) = h_d(n) \quad \text{for} \quad |n| \le 5$$
$$= 0 \quad \text{otherwise}$$

The filter coefficients are symmetrical about $n = 0$ satisfying the condition $(n) = h(-n)$.

For $n = 0$

$$h(0) = \lim_{n\to 0}\left[\frac{\sin \pi n}{\pi n} + \frac{\sin\frac{\pi}{3}n}{\pi n} - \frac{\sin\frac{2\pi}{3}n}{\pi n}\right]$$

$$= \left(1 + \frac{1}{3} - \frac{2}{3}\right) = 0.667$$

$$h(1) = h(-1) = \frac{\sin\pi + \sin\frac{\pi}{3} - \sin\frac{2\pi}{3}}{\pi} = 0$$

$$h(2) = h(-2) = \frac{\sin 2\pi + \sin\frac{2\pi}{3} - \sin\frac{4\pi}{3}}{2\pi} = 0.2757$$

$$h(3) = h(-3) = \frac{\sin 3\pi + \sin\pi - \sin 2\pi}{3\pi} = 0$$

$$h(4) = h(-4) = \frac{\sin 4\pi + \sin\frac{4\pi}{3} - \sin\frac{8\pi}{3}}{4\pi} = -0.1378$$

$$h(5) = h(-5) = \frac{\sin 5\pi + \sin\frac{5\pi}{3} - \sin\frac{10\pi}{3}}{5\pi} = 0$$

The transfer function of the filter is

$$H(z) = h(0) + \sum_{n=1}^{\frac{N-1}{2}}\left[h(n)(z^n + z^{-n})\right]$$

$$= 0.667 + 0.2757(z^2 + z^{-2}) - 0.1378(z^4 + z^{-4}) \qquad (6.67)$$

The transfer function of the realizable filter

$$H'(z) = z^{-5}H(z)$$
$$= -0.1378z^{-1} + 0.2757z^{-3} + 0.667z^{-5} + 0.2757z^{-7} - 0.1378z^{-9} \quad (6.68)$$

The filter coefficients of the causal filter are

$$h(0) = h(10) = h(2) = h(8) = h(4) = h(6) = 0$$
$$h(1) = h(9) = -0.1378$$
$$h(3) = h(7) = 0.2757$$
$$h(5) = 0.667$$

$$\overline{H}(e^{j\omega}) = \sum_{n=0}^{\frac{N-1}{2}} a(n)\cos\omega n$$

$$a(0) = h\left(\frac{N-1}{2}\right) = h(5) = 0.667$$

$$a(n) = 2h\left(\frac{N-1}{2} - n\right)$$

$$a(1) = 2h(5-1) = 2h(4) = 0$$
$$a(2) = 2h(5-2) = 2h(3) = 0.5514$$
$$a(3) = 2h(5-3) = 2h(2) = 0$$
$$a(4) = 2h(5-4) = 2h(1) = -0.2756$$
$$a(5) = 2h(5-5) = 2h(0) = 0$$

$$\overline{H}(e^{j\omega}) = 0.667 + 0.5514\cos 2\omega - 0.2756\cos 4\omega \quad (6.69)$$

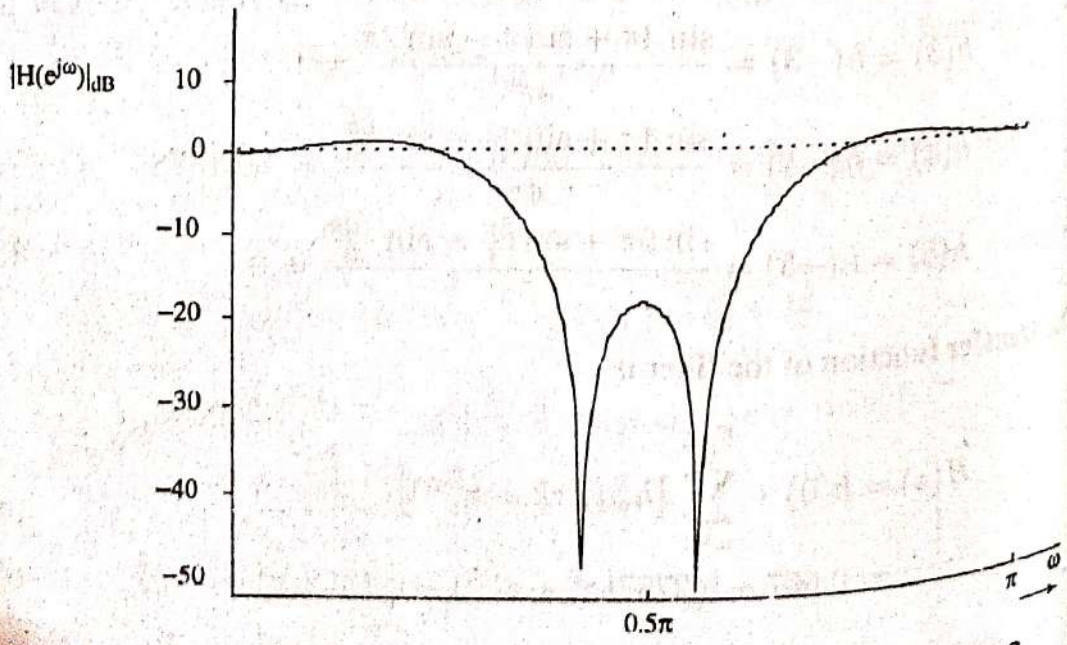


Fig. 6.15 Frequency response of Bandreject filter of example 6.8.

ω (in degrees)	0	15	30	45	60	75
$\overline{H(e^{j\omega})}$	0.9428	1.0067	1.08	0.9426	0.529	0.0516
$\lvert H(e^{j\omega})\rvert_{dB}$	−0.5	0.058	0.67	−0.513	−5.53	−25.7

90	105	120	135	150	165	180
16	0.0516	0.529	0.9426	1.08	1.0067	0.9428
−15.9	−25.7	−5.53	−0.513	0.67	0.058	−0.5

The magnitude vs ω plot of a lowpass filter is shown in Fig. 6.9. We find that there are ripples in the passband and in stopband. These ripples are due to truncation of the infinite series into a finite one. This effect is known as Gibbs phenomenon, discussed in detail in section 6.6.1.

6.6 Design of FIR filters using windows

The desired frequency response $H_d(e^{j\omega})$ of a filter is periodic in frequency and can be expanded in a Fourier series. The resultant series is given by

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d(n)e^{-j\omega n} \tag{6.70}$$

where

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega \tag{6.71}$$

and known as Fourier coefficents having infinite length. One possible way of obtaining FIR filter is to truncate the infinite Fourier series at $n = \pm\left(\frac{N-1}{2}\right)$, where N is the length of the desired sequence. But abrupt truncation of the Fourier series results in oscillation in the passband and stopband. These oscillations are due to slow convergence of the Fourier series and this effect is known as the Gibbs phenomenon. To reduce these oscillations, the Fourier coefficients of the filter are modified by multiplying the infinite impulse response with a finite weighing sequence $w(n)$ called a window where

$$\begin{aligned} w(n) = w(-n) \neq 0 \quad &\text{for} \quad |n| \leq \left(\frac{N-1}{2}\right) \\ = 0 \quad &\text{for} \quad |n| > \left(\frac{N-1}{2}\right) \end{aligned} \tag{6.72}$$

Table 6.2 Filter coefficients of FIR filters.

Type	Coefficients of zero phase filter	Coefficients of linear phase filter with delay $\alpha = \frac{N-1}{2}$
1. Lowpass filter with cutoff frequency ω_c	$h_d(0) = \frac{\omega_c}{\pi}$ $h_d(n) = \frac{\sin \omega_c n}{n\pi} \lvert n \rvert > 0$	$h_d(n) = \frac{\omega_c}{\pi}$ for $n = \alpha$ $= \frac{\sin \omega_c (n-\alpha)}{\pi(n-\alpha)}$ for $n \neq \alpha$
2. Highpass filter with cutoff frequency ω_c	$h_d(0) = 1 - \frac{\omega_c}{\pi}$ $h_d(n) = \frac{-\sin \omega_c n}{n\pi} \lvert n \rvert > 0$	$h_d(n) = 1 - \frac{\omega_c}{\pi}$ for $n = \alpha$ $= \frac{1}{\pi(n-\alpha)}[\sin(n-\alpha)\pi - \sin(n-\alpha)\omega_c] n \neq \alpha$
3. Bandpass filter with cutoff frequencies ω_{c_1} and ω_{c_2}	$h_d(0) = \frac{\omega_{c_2} - \omega_{c_1}}{\pi}$ $h_d(n) = \frac{1}{n\pi}[\sin(\omega_{c_2} n) - \sin(\omega_{c_1} n)] \lvert n \rvert > 0$	$h_d(n) = \frac{\omega_{c_2} - \omega_{c_1}}{\pi}$ for $n = \alpha$ $= \frac{1}{\pi(n-\alpha)}[\sin \omega_{c_2}(n-\alpha) - \sin \omega_{c_1}(n-\alpha)]$
4. Bandreject filter with cutoff frequencies ω_{c_1} and ω_{c_2}	$h_d(0) = 1 - \frac{\omega_{c_2} - \omega_{c_1}}{\pi}$ $= \frac{1}{n\pi}[\sin(\omega_{c_1} n) - \sin(\omega_{c_2} n)]$	$h_d(n) = 1 - \frac{\omega_{c_2} - \omega_{c_1}}{\pi}$ for $n = \alpha$ $= \frac{1}{\pi(n-\alpha)}[\sin \omega_{c_1}(n-\alpha) - \sin \omega_{c_2}(n-\alpha) + \sin(n-\alpha)\pi]$

Example 6.9 Repeat the example 6.6 using (a) Hanning window (b) Hamming window

Solution

(a) Hanning window

$$w_{Hn}(n) = 0.5 + 0.5\cos\frac{2\pi n}{N-1} \quad \text{for } -(N-1)/2 \le n \le (N-1)/2$$
$$= 0 \quad \text{otherwise}$$

For $N = 11$

$$w_{Hn}(n) = 0.5 + 0.5\cos\frac{\pi n}{5} \quad -5 \le n \le 5$$
$$= 0 \quad \text{otherwise}$$
$$w_{Hn}(0) = 0.5 + 0.5 = 1$$
$$w_{Hn}(1) = w_{Hn}(-1) = 0.5 + 0.5\cos\frac{\pi}{5} = 0.9045$$
$$w_{Hn}(2) = w_{Hn}(-2) = 0.5 + 0.5\cos\frac{2\pi}{5} = 0.655$$
$$w_{Hn}(3) = w_{Hn}(-3) = 0.5 + 0.5\cos\frac{3\pi}{5} = 0.345$$
$$w_{Hn}(4) = w_{Hn}(-4) = 0.5 + 0.5\cos\frac{4\pi}{5} = 0.0945$$
$$w_{Hn}(5) = w_{Hn}(-5) = 0.5 + 0.5\cos\pi = 0$$

The filter coefficients can be obtained from the example 6.6, i.e.,

$$h_d(n) = \frac{\sin\pi n - \sin\frac{\pi}{4}n}{n\pi}$$
$$h_d(0) = \lim_{n\to 0}\left[\frac{\sin\pi n - \sin\frac{\pi}{4}n}{\pi n}\right] = 1 - \frac{1}{4} = 0.75$$
$$h_d(-1) = h_d(1) = \frac{\sin\pi - \sin\frac{\pi}{4}}{\pi} = -0.225$$
$$h_d(-2) = h_d(2) = \frac{\sin 2\pi - \sin\frac{\pi}{2}}{2\pi} = -0.159$$
$$h_d(-3) = h_d(3) = \frac{\sin 3\pi - \sin\frac{3\pi}{4}}{3\pi} = -0.075$$
$$h_d(-4) = h_d(4) = \frac{\sin 4\pi - \sin\pi}{4\pi} = 0$$
$$h_d(-5) = h_d(5) = \frac{\sin 5\pi - \sin\frac{5\pi}{4}}{5\pi} = 0.045$$

The filter ... window are

$$h(n) = h_d(n)w_{Hn}(n) \quad \text{for} \quad -5 \le n \le 5$$
$$= 0 \quad \text{otherwise}$$

$$h(0) = h_d(0)w_{Hn}(0) = (0.75)(1) = 0.75$$
$$h(-1) = h(1) = h_d(1)w_{Hn}(1) = (-0.225)(0.905) = -0.204$$
$$h(-2) = h(2) = h_d(2)w_{Hn}(2) = (-0.159)(0.655) = -0.104$$
$$h(-3) = h(3) = h_d(3)w_{Hn}(3) = (-0.075)(0.345) = -0.026$$
$$h(-4) = h(4) = h_d(4)w_{Hn}(4) = (0)(0.8145) = 0$$
$$h(-5) = h(5) = h_d(5)w_{Hn}(5) = (0.045)(0) = 0$$

The transfer function of the filter is given by

$$H(z) = h(0) + \sum_{n=1}^{5} h(n)\left[z^{-n} + z^{n}\right]$$
$$= 0.75 - 0.204\left(z + z^{-1}\right) - 0.104\left(z^2 + z^{-2}\right) - 0.026\left(z^3 + z^{-3}\right)$$

The transfer function of the realizable filter is

$$H'(z) = z^{-5}H(z)$$
$$= -0.026z^{-2} - 0.104z^{-3} - 0.204z^{-4} + 0.75z^{-5} - 0.204z^{-6}$$
$$- 0.104z^{-7} - 0.026z^{-8}$$

The causal filter coefficients are

$$h(0) = h(1) = h(9) = h(10) = 0$$
$$h(2) = h(8) = -0.026$$
$$h(3) = h(7) = -0.104$$
$$h(4) = h(6) = -0.204$$
$$h(5) = 0.75$$

$$\overline{H}(e^{j\omega}) = \sum_{n=0}^{\frac{N-1}{2}} a(n)\cos\omega n$$

$$a(0) = h\left(\frac{N-1}{2}\right) = h(5) = 0.75$$

$$a(n) = 2h\left[\frac{N-1}{2} - n\right]$$

$$a(1) = 2h(5-1) = 2h(4) = -0.408$$
$$a(2) = 2h(5-2) = 2h(3) = -0.208$$
$$a(3) = 2h(5-3) = 2h(2) = -0.052$$
$$a(4) = 2h(5-4) = 2h(1) = 0$$
$$a(5) = 2h(5-5) = 2h(0) = 0$$
$$\overline{H}(e^{j\omega}) = 0.75 - 0.408\cos\omega - 0.208\cos 2\omega - 0.052\cos 3\omega$$

ω (in degrees)	0	15	30	45	60	75	90	105	120	135	150	165	180
$\overline{H}(e^{j\omega})$	0.082	0.139	0.292	0.498	0.702	0.86	0.96	0.999	1.006	1.0017	1	1	1.002
$\lvert H(e^{j\omega})\rvert_{dB}$	−21.72	−17.14	−10.67	−6.05	−3.07	−1.297	−0.3726	−0.0087	0.052	0.015	0	0	0.017

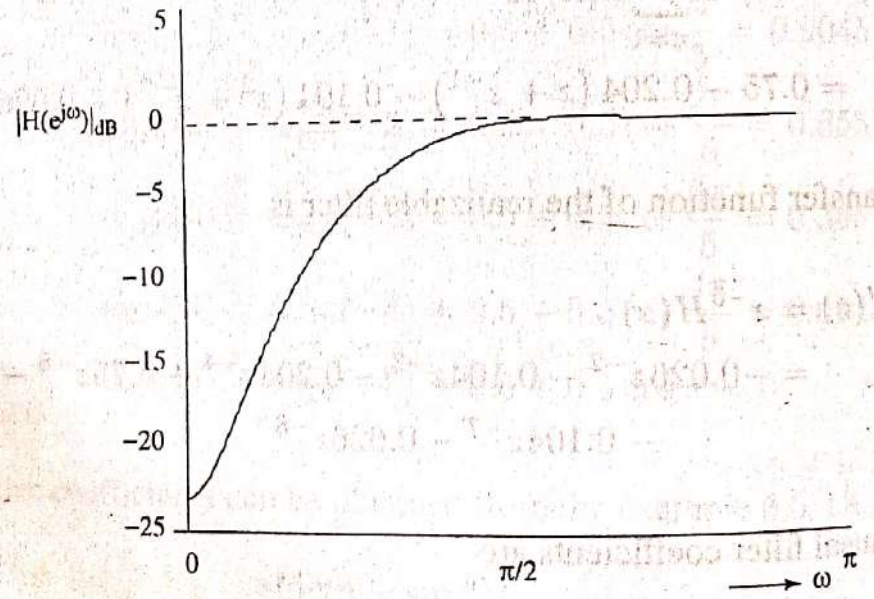


Fig. 6.42 Log magnitude response of example 6.9 using Hanning window.

(b) Hamming window

The Hamming window sequence is given by

$$w_H(n) = 0.54 + 0.46\cos\frac{2\pi n}{N-1} \quad \text{for} \quad -(N-1)/2 \le n \le (N-1)/2$$
$$= 0 \quad \text{otherwise}.$$

The window sequence for $N = 11$ is given by

$$w_H(n) = 0.54 + 0.46\cos\frac{\pi n}{5} \quad \text{for} \quad -5 \le n \le 5$$
$$= 0 \quad \text{otherwise}$$

$$w_H(0) = 0.54 + 0.46 = 1$$

$$w_H(-1) = w_H(1) = 0.54 + 0.46\cos\frac{\pi}{5} = 0.912$$

$$w_H(-2) = w_H(2) = 0.54 + 0.46\cos\frac{2\pi}{5} = 0.682$$

$$w_H(-3) = w_H(3) = 0.54 + 0.46\cos\frac{3\pi}{5} = 0.398$$

$$w_H(-4) = w_H(4) = 0.54 + 0.46\cos\frac{4\pi}{5} = 0.1678$$

$$w_H(-5) = w_H(5) = 0.54 + 0.46\cos\pi = 0.08$$

The filter coefficients using Hamming window sequence are

$$h(n) = h_d(n)w_H(n) \quad \text{for} \quad -5 \le n \le 5$$
$$= 0 \quad \text{otherwise}$$

$$h(0) = h_d(0)w_{Hn}(0) = (1)(0.75) = 0.75$$
$$h(-1) = h(1) = h_d(1)w_{Hn}(1) = (-0.225)(0.912) = -0.2052$$
$$h(-2) = h(2) = h_d(2)w_{Hn}(2) = (-0.159)(0.682) = -0.1084$$
$$h(-3) = h(3) = h_d(3)w_{Hn}(3) = (-0.075)(0.398) = -0.03$$
$$h(-4) = h(4) = h_d(4)w_{Hn}(4) = (0)(0.1678) = 0$$
$$h(-5) = h(5) = h_d(5)w_{Hn}(5) = (-0.045)(0.08) = 0.0036$$

The transfer function of the filter is given by

$$H(z) = h(0) + \sum_{n=1}^{5}\left[h(n)\left(z^{-n} + z^{n}\right)\right]$$
$$= 0.75 - 0.2052\left(z^{-1} + z\right) - 0.1084\left(z^{-2} + z^{2}\right) - 0.03\left(z^{-3} + z^{3}\right)$$
$$+ 0.0036\left(z^{-5} + z^{5}\right)$$

The transfer function of the realizable filter is

$$H'(z) = z^{-5}H(z)$$
$$= 0.0036 - 0.03z^{-2} - 0.1084z^{-3} - 0.2052z^{-4} + 0.75z^{-5}$$
$$- 0.2052z^{-6} - 0.1084z^{-7} - 0.03z^{-8} + 0.0036z^{-10}$$

The filter coefficients of causal filter are

$$h(0) = h(10) = 0.0036; \quad h(1) = h(9) = 0; \quad h(2) = h(8) = -0.03$$
$$h(3) = h(7) = -0.1084; \quad h(4) = h(6) = -0.2052; \quad h(5) = 0.75$$

$$\overline{H}(e^{j\omega}) = \sum_{n=0}^{\frac{N-1}{2}} a(n)\cos\omega n$$

$$a(0) = h\left(\frac{N-1}{2}\right) = h(5) = 0.75$$

$$a(n) = 2h\left[\frac{N-1}{2} - n\right]$$

$$a(1) = 2h(5-1) = 2h(4) = -0.4104$$

$$a(2) = 2h(5-2) = 2h(3) = -0.2168$$

$$a(3) = 2h(5-3) = 2h(2) = -0.06$$

$$a(4) = 2h(5-4) = 2h(1) = 0$$

$$a(5) = 2h(5-5) = 2h(0) = 0.0072$$

$$\overline{H}(e^{j\omega}) = 0.75 - 0.4104\cos\omega - 0.2168\cos 2\omega - 0.06\cos 3\omega + 0.0072\cos 5\omega$$

ω (in degrees)	0	15	30	45	60	75
$\overline{H}(e^{j\omega})$	0.07	0.125	0.28	0.497	0.7168	0.88
$\lvert H(e^{j\omega})\rvert_{dB}$	−23.1	−18	−11	−6.07	−2.89	−1.1

90	105	120	135	150	165	180
0.9668	0.9945	1	1.0026	1.003	1	1.0108
−0.29	−0.0478	0	0.0229	0.028	0	0.093

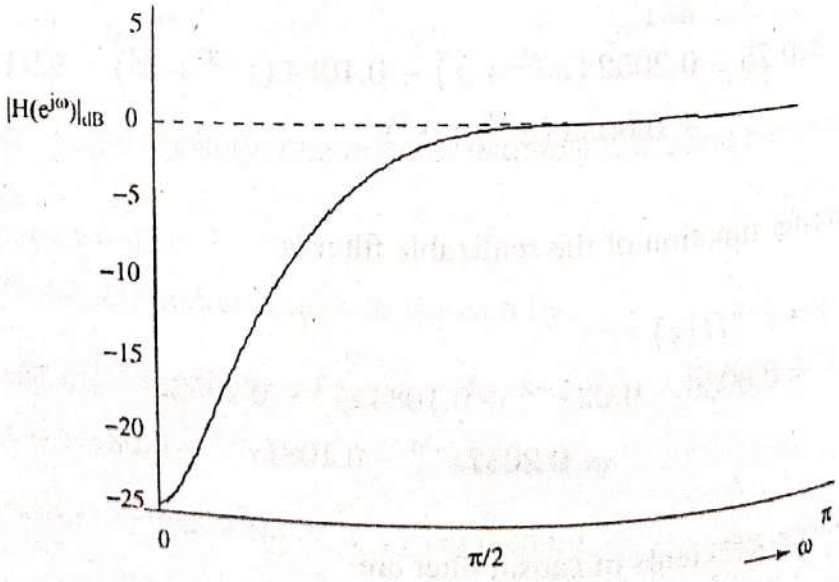


Fig. 6.43 Log magnitude response of Example 6.9 using Hamming window

Example 6.10 Design a filter with

$$H_d(e^{j\omega}) = e^{-j3\omega} \qquad \frac{-\pi}{4} \le \omega \le \frac{\pi}{4}$$
$$= 0 \qquad \frac{\pi}{4} < |\omega| \le \pi$$

Using a Hamming window with $N = 7$

(AU EEE'07)

Solution

Given $H_d(e^{j\omega}) = e^{-j3\omega}$

The frequency response is having a term $e^{-j\omega(N-1)/2}$ which gives $h(n)$ symmetrical about $n = \frac{N-1}{2} = 3$, i.e., we get a causal sequence.

We have

$$h_d(n) = \frac{1}{2\pi}\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} e^{-j3\omega} e^{j\omega n} d\omega$$
$$= \frac{1}{2\pi}\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} e^{j(n-3)\omega} d\omega$$
$$= \frac{\sin\frac{\pi}{4}(n-3)}{\pi(n-3)}$$

For $N = 7$ we have

$$h_d(0) = h_d(6) = 0.075$$
$$h_d(1) = h_d(5) = 0.159$$
$$h_d(2) = h_d(4) = 0.22$$
$$h_d(3) = 0.25$$

The non-causal window sequence is

$$w_{Hn}(n) = 0.5 + 0.5\cos\frac{2\pi n}{N-1} \quad \text{for} \quad -(N-1)/2 \le n \le (N-1)/2$$
$$= 0 \quad \text{otherwise}$$

For $N = 7$

$$w_{Hn}(n) = 0.5 + 0.5\cos\frac{2\pi n}{N-1} \quad \text{for} \quad -3 \le n \le 3$$
$$= 0 \quad \text{otherwise}$$
$$w_{Hn}(0) = 0.5 + 0.5 = 1$$
$$w_{Hn}(-1) = w_{Hn}(1) = 0.5 + 0.5\cos\frac{\pi}{3} = 0.75$$

$$w_{Hn}(-2) = w_{Hn}(2) = 0.5 + 0.5\cos\frac{2\pi}{3} = 0.25$$
$$w_{Hn}(-3) = 0.5 + 0.5\cos\pi = 0$$

The causal window sequence can be obtained by shifting the sequence $w_{Hn}(n)$ to right by 3 samples, i.e.,

$$w_{Hn}(0) = w_{Hn}(6) = 0;\ w_{Hn}(1) = w_{Hn}(5) = 0.25$$
$$w_{Hn}(2) = w_{Hn}(4) = 0.75\ \&\ w_{Hn}(3) = 1$$

The filter coefficients using Hanning window are

$$h(n) = h_d(n)w_{Hn}(n) \quad \text{for} \quad 0 \le n \le 6$$
$$h(0) = h(6) = h_d(0)w_{Hn}(0) = (0.075)(0) = 0$$
$$h(1) = h(5) = h_d(1)w_{Hn}(1) = (0.159)(0.25) = 0.03975$$
$$h(2) = h(4) = h_d(2)w_{Hn}(2) = (0.22)(0.75) = 0.165$$
$$h(3) = h_d(3)w_{Hn}(3) = (0.25)(1) = 0.25$$

Practice Problem 6.4 Design the following filters using Fourier series method. Take $N = 7$

Lowpass filter $H(e^{j\omega}) = 1$ for $0 \le |\omega| \le \frac{\pi}{6}$
$= 0$ otherwise

Highpass filter $H(e^{j\omega}) = 1$ for $\frac{\pi}{6} \le |\omega| \le \pi$
$= 0$ otherwise

Bandpass filter $H(e^{j\omega}) = 1$ for $\frac{\pi}{6} \le |\omega| \le \frac{\pi}{3}$
$= 0$ otherwise

Bandstop filter $H(e^{j\omega}) = 1$ for $0 \le |\omega| \le \frac{\pi}{6}$ and $\frac{\pi}{3} \le \omega \le \pi$
$= 0$ otherwise.

Practice Problem 6.5 Design a filter with

$$H_d(e^{j\omega}) = e^{-j5\omega} \quad -\frac{\pi}{2} \le \omega \le \frac{\pi}{2}$$
$$= 0 \quad \frac{\pi}{2} < |\omega| \le \pi$$

using Blackman window with $N = 11$.

Example 6.11 Design an FIR lowpass filter satisfying the following specifications

$\alpha_p \leq 0.1$ dB $\quad \alpha_s \geq 44.0$ dB

$\omega_p = 20$ rad/sec; $\quad \omega_s = 30$ rad/sec; $\quad \omega_{sf} = 100$ rad/sec

Solution

From the given specifications

$$B = \omega_s - \omega_p = 10 \text{ rad/sec}$$

$$\omega_c = \frac{1}{2}(\omega_p + \omega_s) = 25 \text{ rad/sec}$$

$$\omega_c \text{ (in radians)} = \omega_c T = \omega_c \cdot \frac{2\pi}{\omega_{sf}} = (25)\frac{(2\pi)}{100} = \frac{\pi}{2}$$

Step 1.

$$H(e^{j\omega}) = 1 \quad \text{for } |\omega| \leq \frac{\pi}{2}$$

$$= 0 \quad \text{for } \frac{\pi}{2} < |\omega| \leq 2\pi$$

$$h_d(n) = \frac{1}{2\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} e^{j\omega n} d\omega = \frac{\sin\frac{\pi}{2}n}{\pi n}$$

Step 2.

$$\delta_1 = 10^{-0.05(44)} = 6.3096 \times 10^{-3}$$

$$\delta_2 = \frac{10^{+0.05(0.1)} - 1}{10^{+0.05(0.1)} + 1} = 5.7563 \times 10^{-3}$$

$$\delta = \min(\delta_1, \delta_2)$$

$$= 5.7563 \times 10^{-3}$$

Step 3. $\alpha_s = -20\log_{10}\delta = 44.797$ dB.

Step 4. From Eq. (6.100) for $\alpha_s = 44.797$ dB

$$\alpha = 0.5842(\alpha_s - 21)^{0.4} + 0.07886(\alpha_s - 21)$$
$$= 3.9524$$

Step 5. From Eq. (6.101) for $\alpha_s = 44.797$ dB

$$D = \frac{\alpha_s - 7.95}{14.36} = 2.566$$

Step 6.

$$N \geq \frac{\omega_{sf} D}{B} + 1$$
$$\geq \frac{100(2.566)}{10} + 1$$
$$\geq 26.66$$

Hence, $N = 27$.

Step 7. The window sequence $\omega_k(n) = \dfrac{I_0\left[\alpha\sqrt{1-\left(\frac{2n}{N-1}\right)^2}\right]}{I_0(\alpha)}$ for $|n| \leq \frac{N-1}{2}$,

$\alpha = 3.9524$, from Eq. (6.90) we can find

$$I_0(\alpha) = 10.8468$$

$$w_k(0) = \frac{I_0(\alpha)}{I_0(\alpha)} = 1$$

$$w_k(1) = \omega_k(-1) = \frac{I_0(3.9406)}{10.8468} = \frac{10.7379}{10.8468} = 0.9899$$

$$w_k(2) = \omega_k(-2) = \frac{I_0(3.9073)}{10.8468} = \frac{10.4163}{10.8468} = 0.9603$$

$$w_k(3) = \omega_k(-3) = \frac{I_0(3.8457)}{10.8468} = \frac{9.8964}{10.8468} = 0.9124$$

$$w_k(4) = \omega_k(-4) = \frac{I_0(3.7606)}{10.8468} = \frac{9.2018}{10.8468} = 0.84835$$

$$w_k(5) = \omega_k(-5) = \frac{I_0(3.623)}{10.8468} = \frac{8.1856}{10.8468} = 0.75465$$

$$w_k(6) = \omega_k(-6) = \frac{I_0(3.5062)}{10.8468} = \frac{7.4168}{10.8468} = 0.68378$$

$$w_k(7) = \omega_k(-7) = \frac{I_0(3.3305)}{10.8468} = \frac{6.4025}{10.8468} = 0.59027$$

$$w_k(8) = \omega_k(-8) = \frac{I_0(3.1154)}{10.8468} = \frac{5.3613}{10.8468} = 0.49428$$

$$w_k(9) = \omega_k(-9) = \frac{I_0(2.852)}{10.8468} = \frac{4.3331}{10.8468} = 0.3995$$

$$w_k(10) = \omega_k(-10) = \frac{I_0(2.5257)}{10.8468} = \frac{3.3553}{10.8468} = 0.30934$$

$$w_k(11) = \omega_k(-11) = \frac{I_0(2.1063)}{10.8468} = \frac{2.4574}{10.8468} = 0.2265$$

$$w_k(12) = \omega_k(-12) = \frac{I_0(1.52)}{10.8468} = \frac{1.6666}{10.8468} = 0.1536$$

$$w_k(13) = \omega_k(-13) = \frac{I_0(0)}{10.8468} = \frac{1}{10.8468} = 0.0922$$

In all the above cases, to calculate $I_0(x)$, the sum of first seven terms in the series is considered.

The impulse response $h_d(n)$ and $h(n)$ are given in table 6.5.

Table 6.5

n	$h_d(n)$	$h(n) = h_d(n)\omega_k(n)$
0	0.5	0.5
1	0.318	0.31479
2	0	0
3	-0.106	-0.0967
4	0	0
5	0.06366	0.04804
6	0	0
7	-0.0454	-0.0268
8	0	0
9	0.03536	0.014126
10	0	0
11	-0.0289	-0.006546
12	0	0
13	0.02448	0.002267

The transfer function is given by

$$H(z) = z^{-13}\left[h(0) + \sum_{n=0}^{13} h(n)(z^n + z^{-n})\right]$$

Example 6.12 Design a FIR bandpass digital filter satisfying the following specifications

$f_{p1} = 20\,\text{Hz}$ $\qquad$ $\alpha_p = 0.5\,\text{dB}$

$f_{p2} = 30\,\text{Hz}$ $\qquad$ $\alpha_s = 30\,\text{dB}$

$$f_{s1} = 10\,\text{Hz}$$
$$f_{s2} = 40\,\text{Hz}$$
$$F = 100\,\text{Hz}$$

Solution

$$\omega_{p1} = 40\pi \text{ rad/sec} \qquad \omega_{s1} = 20\pi \text{ rad/sec}$$
$$\omega_{p2} = 60\pi \text{ rad/sec} \qquad \omega_{s2} = 80\pi \text{ rad/sec}$$
$$\omega_{sf} = 200\pi \text{ rad/sec}$$

$$\begin{aligned} B &= \min[\omega_{p1} - \omega_{s1}, \omega_{s2} - \omega_{p2}] \\ &= \min[20\pi, 20\pi] \\ &= 20\pi \quad \text{rad/sec} \end{aligned}$$

$$\omega_{c1} = \omega_{p1} - \frac{B}{2} = 40\pi - \frac{20\pi}{2} = 30\pi \text{ rad/sec}$$

$$\omega_{c2} = \omega_{p2} + \frac{B}{2} = 60\pi + \frac{20\pi}{2} = 70\pi \text{ rad/sec}$$

ω_{c1} and ω_{c2} in radians are given by

$$\omega_{c1} = 30\pi \times \frac{2\pi}{200\pi} = 0.3\pi$$
$$\omega_{c2} = 70\pi \times \frac{2\pi}{200\pi} = 0.7\pi$$

Step 1

$$\begin{aligned} H(e^{j\omega}) &= 1 \quad \text{for} \quad 0.3\pi \le |\omega| \le 0.7\pi \\ &= 0 \qquad\quad \text{otherwise} \end{aligned}$$

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi}\left[\int_{-0.7\pi}^{-0.3\pi} e^{j\omega n} d\omega + \int_{0.3\pi}^{0.7\pi} e^{j\omega n} d\omega\right] \\ &= \frac{\sin 0.7\pi n - \sin 0.3\pi n}{n\pi} \end{aligned}$$

Step 2

$$\delta_1 = 10^{-0.05(30)} = 0.0316$$
$$\delta_2 = \frac{10^{+0.05(0.5)-1}}{10^{+0.05(0.5)+1}} = 0.0287$$
$$\begin{aligned} \delta &= \min(\delta_1, \delta_2) \\ &= 0.0287 \end{aligned}$$

Step 3

Step 4 From Eq.(6.100) for $\alpha_s = 30.84$ dB

$$\begin{aligned}\alpha &= 0.5842(\alpha_s - 21)^{0.4} + 0.07886(\alpha_s - 21) \\ &= 2.23413\end{aligned}$$

Step 5 From Eq.(6.101) for $\alpha_s = 30.84$ dB

$$D = \frac{\alpha_s - 7.95}{14.36} = 1.594$$

Step 6

$$\begin{aligned}N &\geq \frac{\omega_{sf} D}{B} + 1 \\ N &\geq \frac{(200\pi)(1.594)}{20\pi} + 1 \\ &\geq 16.94\end{aligned}$$

Hence $N = 17$.

Step 7 The window sequence

$$w_k(n) = \frac{I_o \alpha \sqrt{1 - (2n/N - 1)^2}}{I_0(\alpha)} \quad \text{for} \quad |n| \leq \frac{N-1}{2}$$

For $\alpha = 2.23413$

$$I_0(\alpha) = 2.6955$$

$$w_k(0) = \frac{I_o(\alpha)}{I_0(\alpha)} = 1$$

$$w_k(1) = w_k(-1) = \frac{I_0(2.2166)}{2.6955} = \frac{2.6617}{2.6955} = 0.98726$$

$$w_k(2) = w_k(-2) = \frac{I_0(2.1632)}{2.6955} = \frac{2.5598}{2.6955} = 0.9497$$

$$w_k(3) = w_k(-3) = \frac{I_0(2.0711)}{2.6955} = \frac{2.3965}{2.6955} = 0.88907$$

$$w_k(4) = w_k(-4) = \frac{I_0(1.9348)}{2.6955} = \frac{2.1790}{2.6955} = 0.80837$$

$$w_k(5) = w_k(-5) = \frac{I_0(1.744)}{2.6955} = \frac{1.91774}{2.6955} = 0.71146$$

$$w_k(6) = w_k(-6) = \frac{I_0(1.4777)}{2.6955} = \frac{1.6251}{2.6955} = 0.6029$$

$$w_k(7) = w_k(-7) = \frac{I_0(1.0816)}{2.6955} = \frac{1.3145}{2.6955} = 0.4877$$

$$w_k(8) = w_k(-8) = \frac{I_0}{2.6955} = \frac{1}{2.6955} = 0.37099$$

The impulse response $h_d(n)$ and $h(n)$ are in table 6.6

Table 6.6

n	$h_d(n)$	$h(n) = h_d(n)w_k(n)$
0	0.4	0.4
1	0	0
2	-0.3027	-0.28747
3	0	0
4	0.09354	0.0756
5	0	0
6	0.062	0.0373
7	0	0
8	-0.0757	-0.02808

The transfer function is given by

$$H(z) = z^{-8}\left[h(0) + \sum_{n=0}^{8} h(n)(z^n + z^{-n})\right]$$

Practice Problem 6.6 Design an FIR lowpass filter satisfying the following specifications

$\alpha_p \leq 0.5\,\text{dB}$ $\quad$ $\alpha_s \geq 31\,\text{dB}$

$\omega_p = 10\,\text{rad/sec}$ $\quad$ $\omega_s = 25\,\text{rad/sec}$

$\omega_{sf} = 100\,\text{rad/sec.}$

6.6.8 *Summary of windows*

The different windows parameters are compared in table 6.3. Looking at the parameters for rectangular and triangular window

[Finite Impulse Response].

* An FIR filter is a filter whose impulse response is finite. The design of FIR filter consists of all the finite no. of sample of impulse response.

* The response of FIR filter depends on present & past inputs but not on past outputs.

* The general form of difference equation of an FIR filter is described by,

$$y(n) = \sum_{k=0}^{M} b_k x(n-k)$$

$$Y(z) = \sum_{k=0}^{M} b_k . z^{-k} X(z)$$

$$\frac{Y(z)}{X(z)} = \sum_{k=0}^{M} b_k . z^{-k}$$

$$H(z) = \sum_{k=0}^{M} b_k . z^{-k}$$

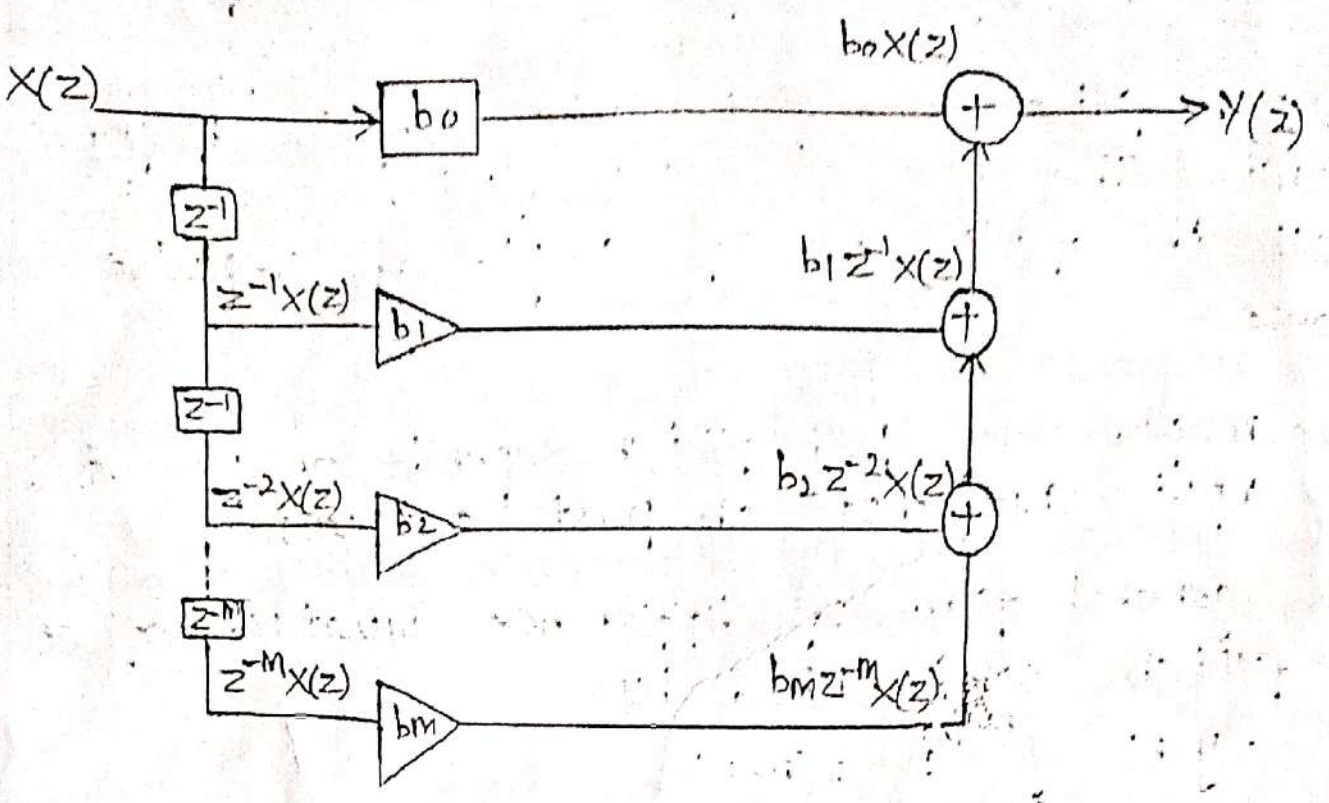
* The different structures used for the realization of FIR filters are:

1) Direct form realization structure.
2) Cascade form realization structure.
3) Linear phase realization structure.

1. Direct Form Realization Structure:-

It is the direct implementation of an FIR filter described by the difference equation:

$$Y(z) = \sum_{k=0}^{M} b_k . z^{-k} X(z)$$



⇒ Realize the FIR system described by the transfer function $H(z) = 1 + \frac{1}{5}z^{-1} + \frac{3}{4}z^{-2} + \frac{1}{3}z^{-3} + \frac{1}{7}z^{-4} + \frac{1}{6}z^{-5}$

A. Given,

$$H(z) = 1 + \frac{1}{5}z^{-1} + \frac{3}{4}z^{-2} + \frac{1}{3}z^{-3} + \frac{1}{7}z^{-4} + \frac{1}{6}z^{-5}$$

$$H(z) = \frac{Y(z)}{X(z)}$$

$$\frac{Y(z)}{X(z)} = 1 + \frac{1}{5}z^{-1} + \frac{3}{4}z^{-2} + \frac{1}{3}z^{-3} + \frac{1}{7}z^{-4} + \frac{1}{6}z^{-5}$$

$$Y(z) = X(z) + \frac{1}{5}z^{-1} + \frac{3}{4}z^{-2} + \frac{1}{3}z^{-3} + \frac{1}{7}z^{-4} + \frac{1}{6}z^{-5}$$

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=> Realize the FIR filter described by the transfer function $H(z) = (1+\frac{1}{2}z^{-1}+z^{-2})(1+\frac{1}{5}z^{-1}+\frac{1}{3}z^{-2})$.

A: Given,

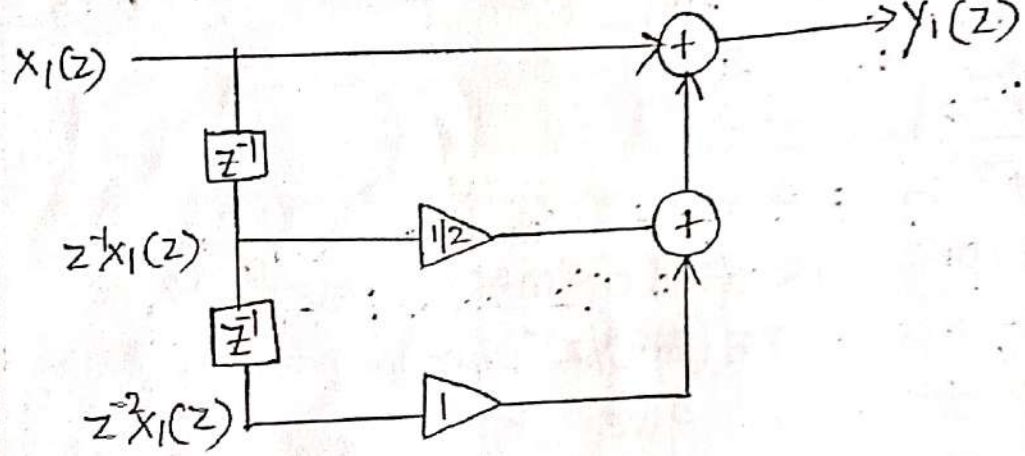
$$H(z) = (1+\tfrac{1}{2}z^{-1}+z^{-2})(1+\tfrac{1}{5}z^{-1}+\tfrac{1}{3}z^{-2})$$

$$H(z) = H_1(z).H_2(z)$$

$$H_1(z) = (1+\tfrac{1}{2}z^{-1}+z^{-2})$$

$$\frac{Y_1(z)}{X_1(z)} = 1+\tfrac{1}{2}z^{-1}+z^{-2}$$

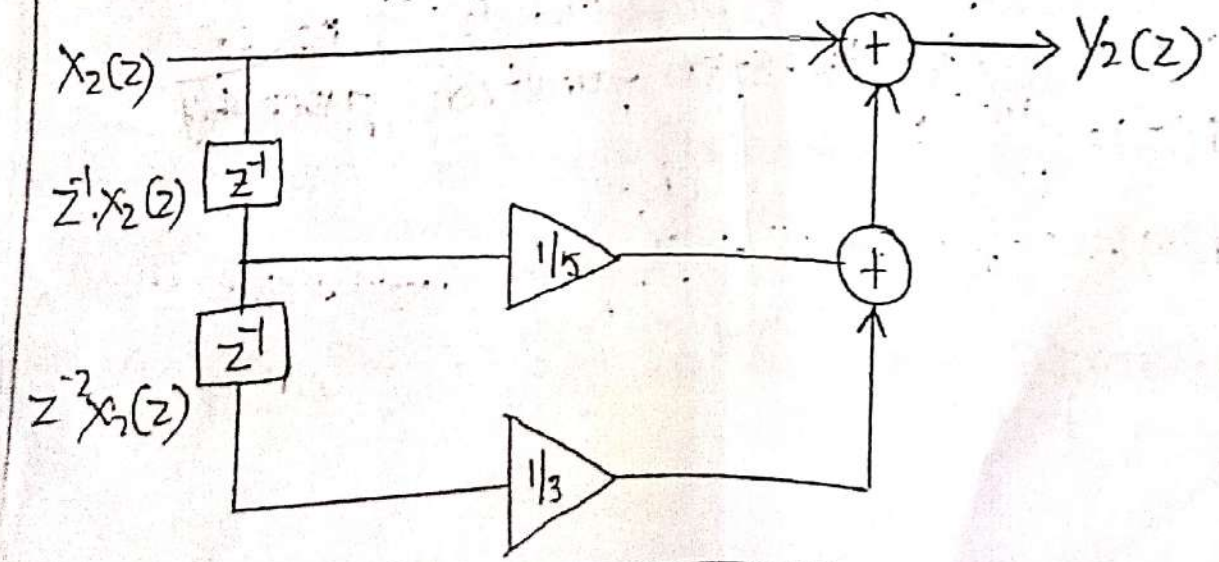
$$Y_1(z) = X_1(z) + \tfrac{1}{2}z^{-1}X_1(z) + z^{-2}X_1(z)$$



Ily: $H_2(z) = 1+\frac{1}{5}z^{-1}+\frac{1}{3}z^{-2}$

$$\frac{Y_2(z)}{X_2(z)} = 1+\tfrac{1}{5}z^{-1}+\tfrac{1}{3}z^{-2}$$

$$Y_2(z) = X_2(z) + \tfrac{1}{5}z^{-1}X_2(z) + \tfrac{1}{3}z^{-2}X_2(z)$$



2. Cascade form structure:-

In cascade structure the given transfer function H(z) can be expressed as the product of no. of sub-transfer functions. Where, each of the sub-transfer function is realized by using direct form structure and all the sub-transfer functions realized are interconnected in series are cascade.

The transfer function of FIR filter which is of order (M-1) is given by:

$$H(z) = \sum_{k=0}^{M-1} b_k \cdot z^{-k}$$

If the order of FIR filter (M-1) is even, then H(z) can be expressed as (M-1)/2 second ordered sub-transfer function is given by:

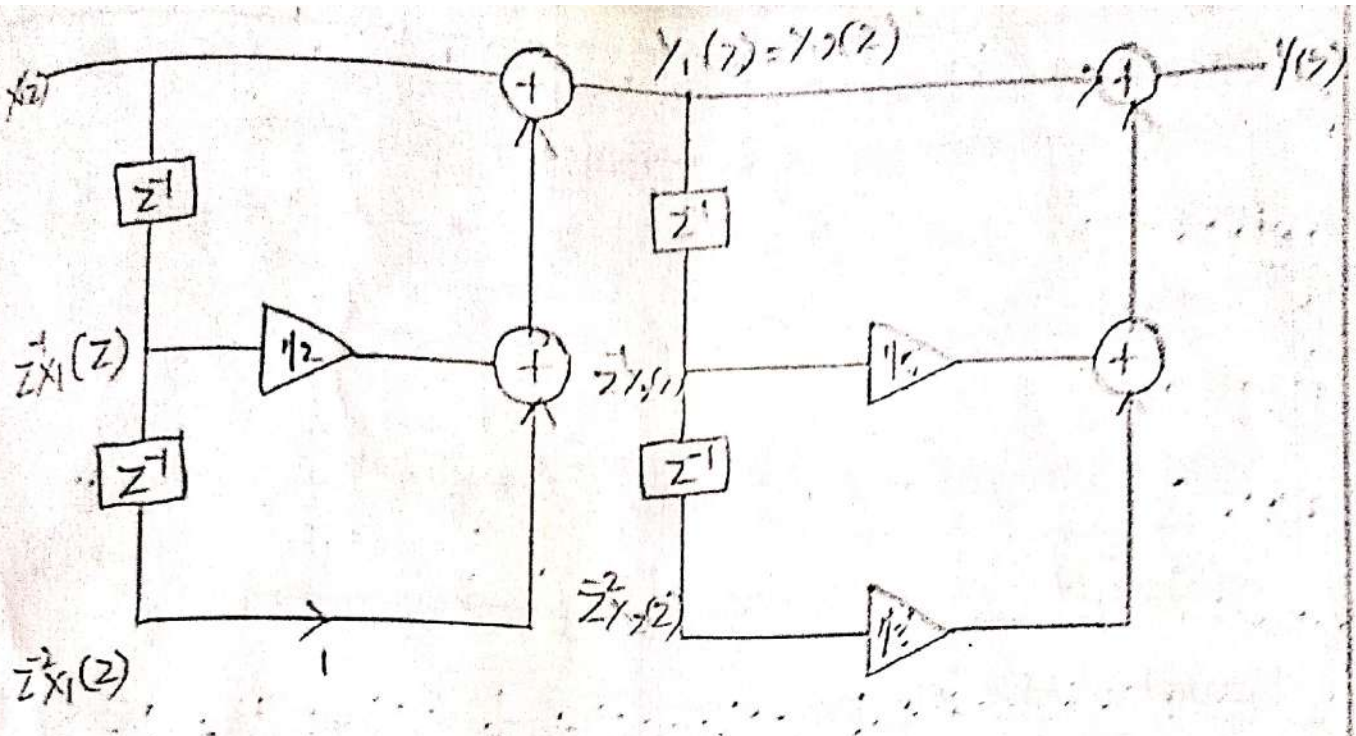
$$H(z) = \sum_{k=0}^{M-1} b_k \cdot z^{-k}$$

$$H(z) = \prod^{\left(\frac{M-1}{2}\right)} (C_{0i} + C_{1i} z^{-1} + C_{2i} z^{-2})$$

If the order of FIR filter (M-1) is odd, then H(z) can be expressed as a product of one. First order sub-transfer function and $\left(\frac{M-2}{2}\right)$ no. of second orders sub-transfer function given by:

$$H(z) = \sum_{k=0}^{M-1} b_k \cdot z^{-k}$$

$$H(z) = (C_{0i} + C_{1i} z^{-1}) + \prod_{i=1}^{\left(\frac{M-2}{2}\right)} (C_{0i} + C_{1i} z^{-1} + C_{2i} z^{-2}).$$



3. Linear phase Realization Structure:-

If the impulse of FIR filter is h(k), then the transfer function of FIR filter is given by,

$$H(z) = \sum_{k=0}^{M-1} h(k).\, z^{-k}$$

Where,

M = length of sequence.

(M-1) = Order of sequence

An FIR filter is said to be linear phase if it satisfies the condition;

$$h(k) = \pm h(M-1-k)$$

An FIR filter is said to be symmetric if,

$$h(k) = +h(M-1-k)$$

And the FIR filter is said to be anti-symmetric if,

$$h(k) = -h(M-1-k).$$

2. Obtain the Linear phase realization structure for FIR filter given by the transfer function.

$H(z) = \frac{1}{8} + \frac{1}{5}z^{-1} + \frac{1}{5}z^{-2} + \frac{1}{8}z^{-3}$

Given,

$H(z) = \frac{1}{8} + \frac{1}{5}z^{-1} + \frac{1}{5}z^{-2} + \frac{1}{8}z^{-3} \longrightarrow (1)$

$h(k) = \{\frac{1}{8}, \frac{1}{5}, \frac{1}{5}, \frac{1}{8}\} \longrightarrow$ symmetric

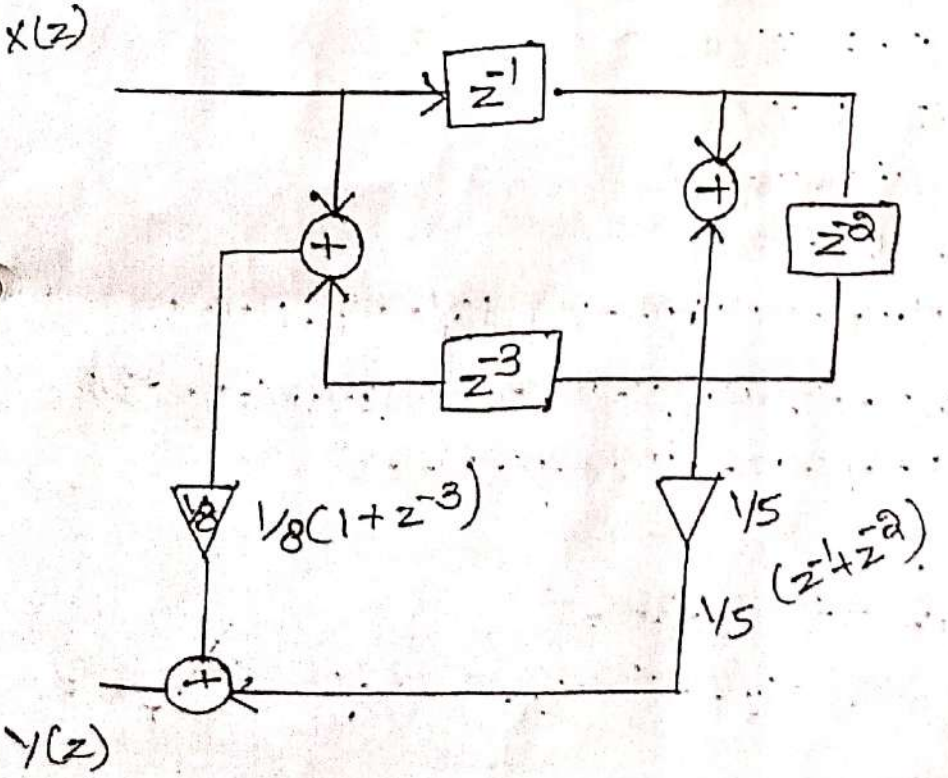
Rewrite the eq (1),

$H(z) = \frac{1}{8}(1+z^{-3}) + \frac{1}{5}(z^{-1}+z^{-2}) \longrightarrow (2)$

No. of multipliers required $= \frac{N+1}{2} = \frac{4+1}{2} = \frac{5}{2} \simeq 3$.

$\Rightarrow$ When $N=4$ (even)

No. of delay elements are 3.



5. Realize the linear phase structure of FIR filter for the described transfer function.

$H(z) = 1 + \frac{1}{3}z^{-1} + \frac{1}{4}z^{-2} + \frac{1}{4}z^{-3} + \frac{1}{3}z^{-4} + z^{-5}$

A. Given function,

$H(z) = 1 + \frac{1}{3}z^{-1} + \frac{1}{4}z^{-2} + \frac{1}{4}z^{-3} + \frac{1}{3}z^{-4} + z^{-5}$ ⟶ (1)

We know that for FIR filter

$$H(z) = \sum_{k=0}^{M-1} h(k).z^{-k}$$

$$H(z) = \sum_{k=0}^{5} h(k).z^{-k}$$

$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5}$ ⟶ (2)

On comparing eq (1) & eq (2) we get,

$h(0)=1,\ h(1)=1/3,\ h(2)=1/4,\ h(3)=1/4,\ h(4)=1/3,\ h(5)=1$

Condition for symmetric is,

$h(k) = h(M-1-k)$

$h(0) = h(5-0) = h(5)$

$h(1) = h(5-1) = h(4)$

$h(2) = h(5-2) = h(3)$

The condition is satisfied.

From eq (1) we can write

$\frac{Y(z)}{X(z)} = 1 + 1/3\,z^{-1} + 1/4\,z^{-2} + 1/4\,z^{-3} + 1/3\,z^{-4} + z^{-5}$

$Y(z) = X(z) + 1/3\,X(z)\,z^{-1} + 1/4\,X(z).z^{-2} + 1/4\,X(z)\,z^{-3} + 1/3\,X(z).z^{-4} + X(z).z^{-5}$

When, $N = 6 = \text{even}$,

No. of delay elements are 5.

$Y(z) = X(z)\left(1 + \frac{1}{3}z^{-1} + 1/4.z^{-2} + 1/4\,z^{-3} + 1/3\,z^{-4} + z^{-5}\right)$.

$\frac{Y(z)}{X(z)} = 1 + \frac{1}{3}z^{-1} + \frac{1}{4}z^{-2} + \frac{1}{4}z^{-3} + \frac{1}{3}z^{-4} + z^{-5}$

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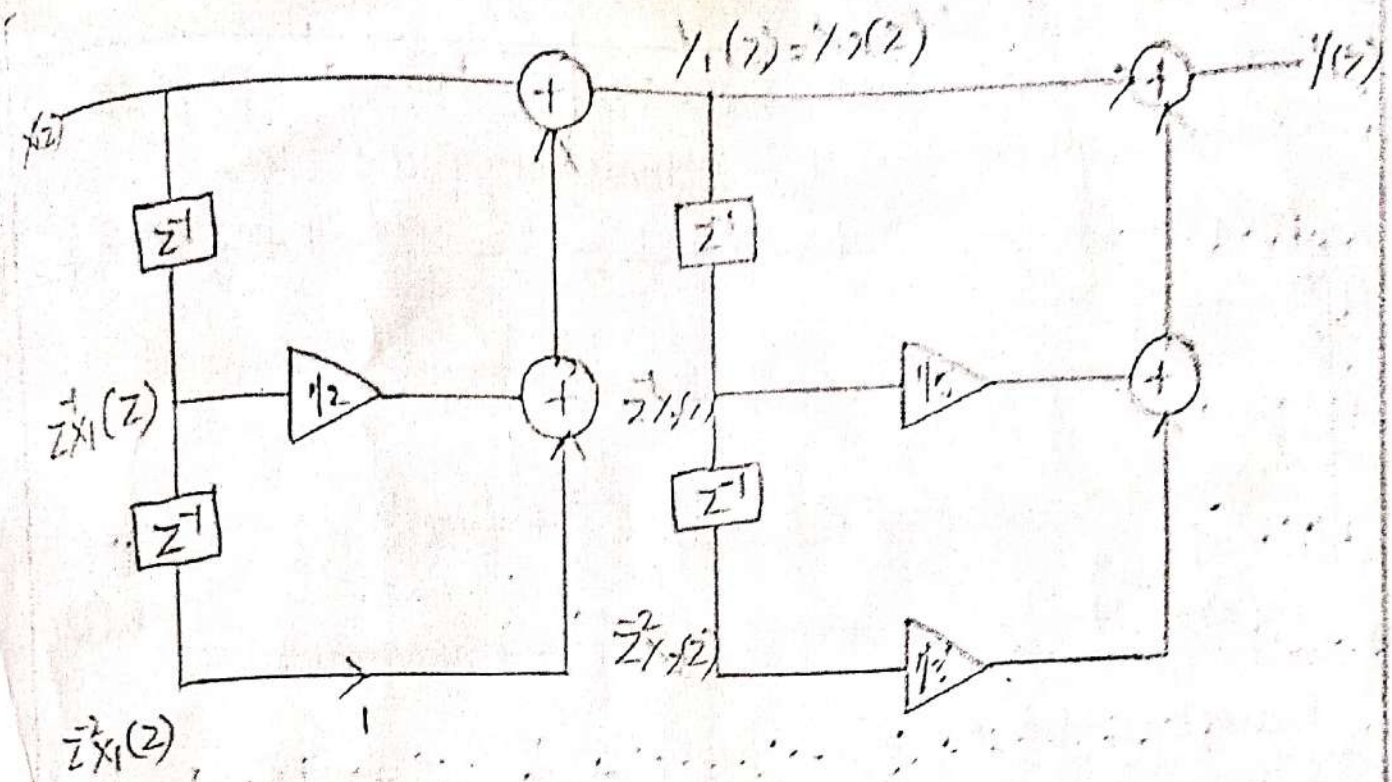


3. Linear phase Realization Structure:

If the impulse of FIR filter is h(k), then the transfer function of FIR filter is given by,

$$H(z) = \sum_{k=0}^{M-1} h(k) \cdot z^{-k}$$

Where,

M = length of sequence.

(M-1) = Order of sequence

An FIR filter is said to be linear phase if it satisfies the condition;

$$h(k) = \pm h(M-1-k)$$

An FIR filter is said to be symmetric if,

$$h(k) = +h(M-1-k)$$

And the FIR filter is said to be anti-symmetric if,

$$h(k) = -h(M-1-k).$$

1. Obtain the linear phase realization of FIR filter given by the transfer function.

$H(z) = 1 + \frac{1}{3}z^{-1} + \frac{1}{8}z^{-2} + \frac{1}{4}z^{-3} + \frac{1}{8}z^{-4} + \frac{1}{3}z^{-5} + z^{-6}$

A. Method - I :-

Given,

$H(z) = 1 + \frac{1}{3}z^{-1} + \frac{1}{8}z^{-2} + \frac{1}{4}z^{-3} + \frac{1}{8}z^{-4} + \frac{1}{3}z^{-5} + z^{-6}$ ↳ (1)

$h(k) = \{1 + \frac{1}{3}, \frac{1}{8}, \frac{1}{4}, \frac{1}{8}, \frac{1}{3}, 1\} \longrightarrow$ symmetric.

Rewrite the eq (1),

$H(z) = 1(z^0 + z^{-6}) + \frac{1}{3}(z^{-1} + z^{-5}) + \frac{1}{8}(z^{-2} + z^{-4}) + \frac{1}{4}z^{-3}$

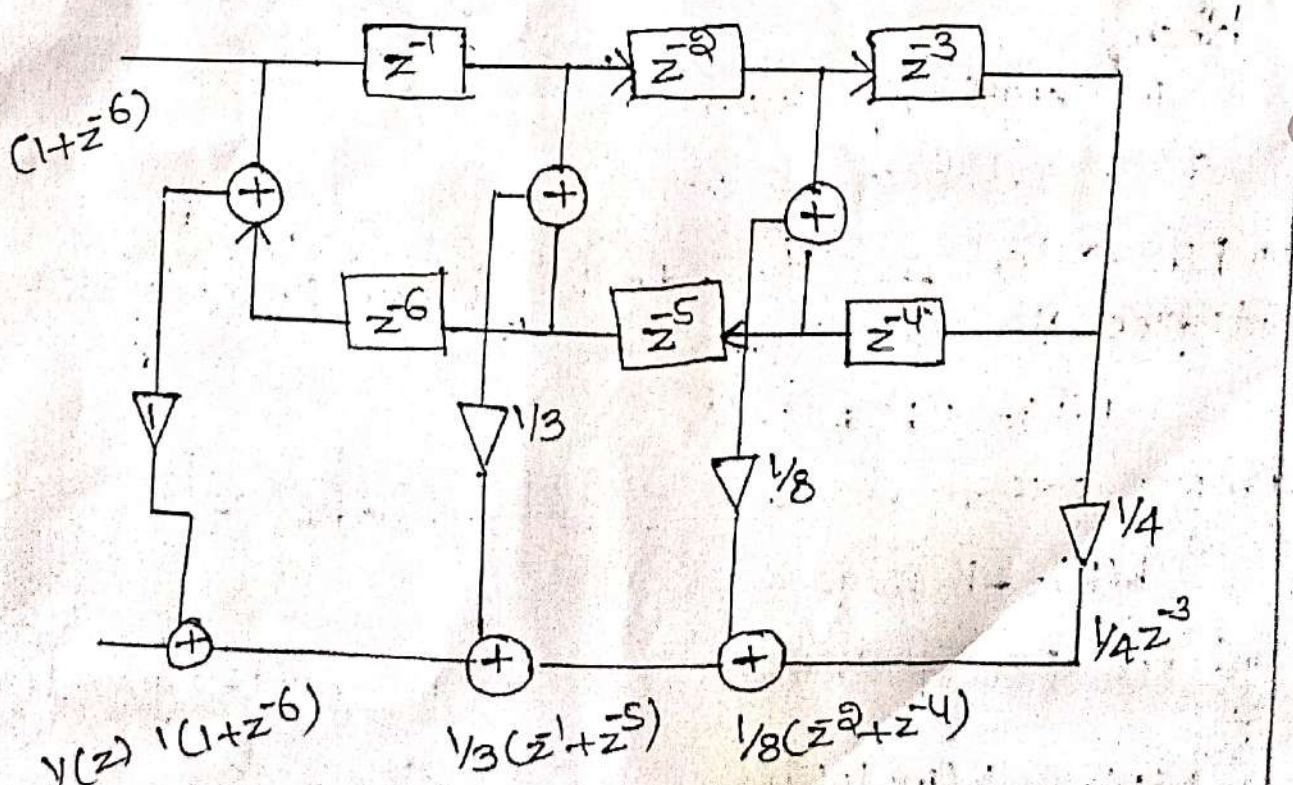
↳ eq (2).

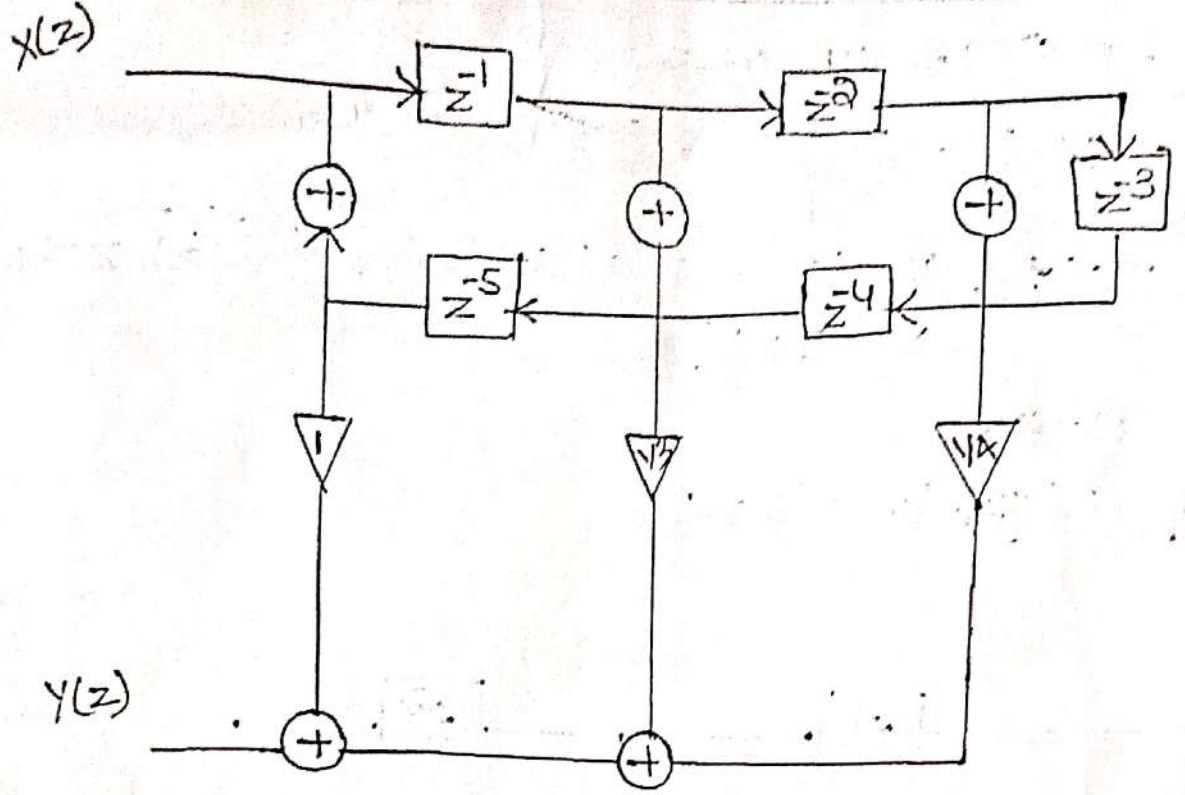
No. of multipliers required $= \frac{N+1}{2} = \frac{7+1}{2} = \frac{8}{2} = 4$.

Case - (i) :-

When N = 7 (Odd)

No. of delay elements required = 6.





$H(z) = (1+z^{-1})(1+1/2\,z^{-1}+1/2\,z^{-2}+z^{-3})$

Given, $H(z) = (1+z^{-1})(1+1/2\,z^{-1}+1/2\,z^{-2}+z^{-3})$

$H(z) = 1+\frac{3}{2}z^{-1}+z^{-2}+3/2\,z^{-3}+z^{-4} \longrightarrow (1)$.

We know that for FIR filter

$$H(z) = \sum_{k=0}^{M-1} h(k).z^{-k}$$

$$H(z) = \sum_{k=0}^{4} h(k).z^{-k}$$

$H(z) = h(0)+h(1)\,z^{-1}+h(2)z^{-2}+h(3)\,z^{-3}+h(4)\,z^{-4} \longrightarrow (2)$.

On comparing eq (1) & eq (2), we get.

$h(0)=1,\ h(1)=3/2,\ h(2)=1\ \ h(3)=3/2,\ h(4)=1$

Condition for symmetric is,

$h(K) = h(M-1-K)$

$h(0) = h(4-0) = h(4)$

$h(1) = h(4-1) = h(3)$

$h(2) = h(4-2) = h(2)$.

The conditions is satisfied.

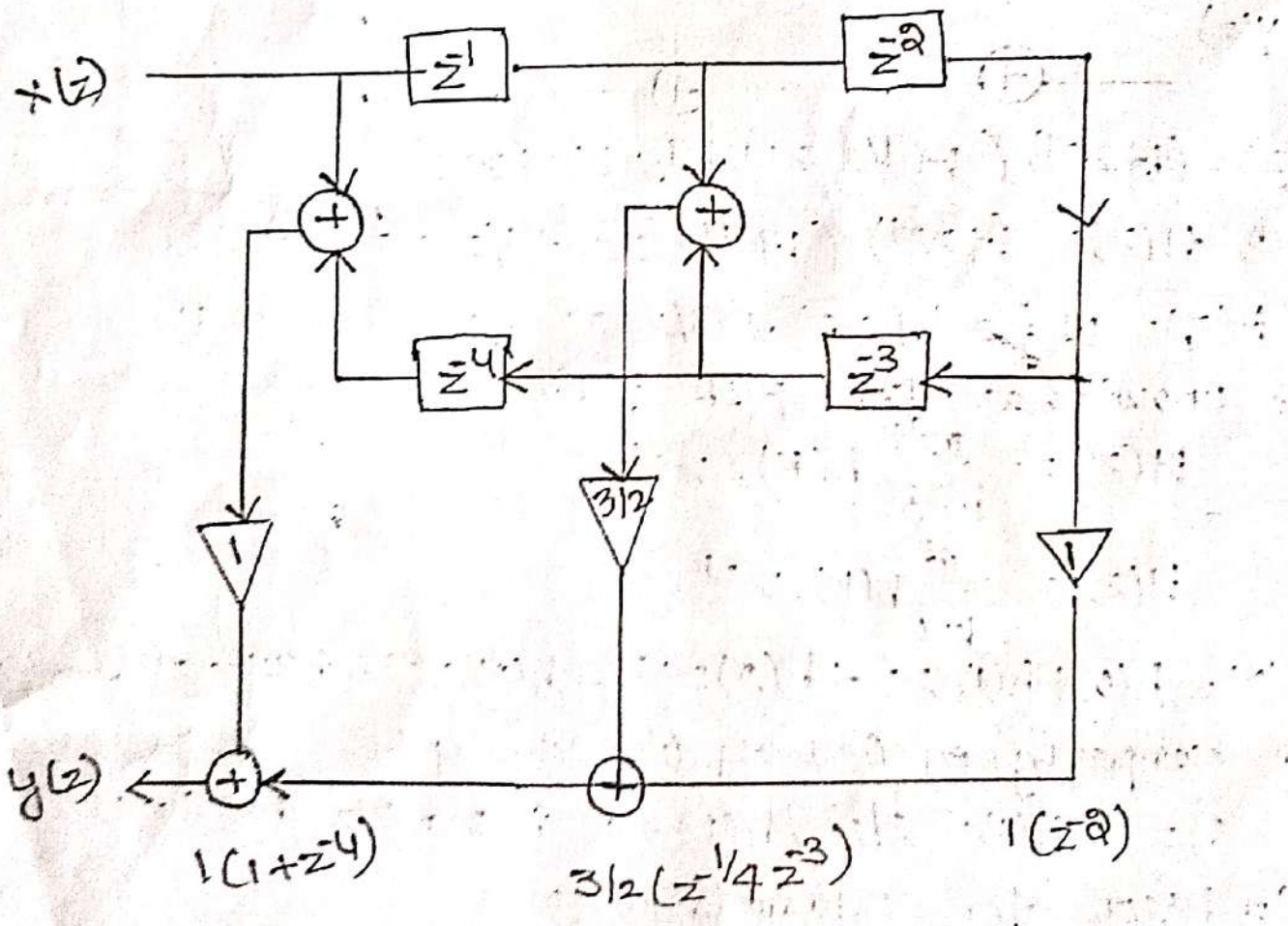
From eq (1) we can write.

$$\frac{Y(z)}{X(z)} = 1 + \frac{3}{2} z^{-1} + z^{-2} + \frac{3}{2} z^{-3} + z^{-4}$$

$$Y(z) = X(z) + \frac{3}{2} X(z) z^{-1} + X(z) z^{-2} + \frac{3}{2} X(z) \cdot z^{-3} + X(z) \cdot z^{-4}$$

When $N = 4$.

No. of delay elements = 3.



storage and retrial of information.

11.3 TMS320C5x Family of Digital Signal Processors

The TMS320C5x family of processors are fifth-generation digital signal processors from Texas Instruments, USA. They are 16-bit fixed point processors fabricated using high performance static CMOS technology. These processors have advanced Harvard architecture, with a variety of on-chip peripherals and memory and highly specialized instructions. They can execute 50 Million Instructions Per Second(MIPS).

Some of the features of TMS320C5x family of digital signal processors are,

- 16-bit CPU
- 20 to 50 ns single cycle instruction execution time
- Single cycle 16 × 16-bit MAC (Multiply/Accumulate) unit
- 64k × 16-bit external program memory address space
- 64k × 16-bit external data memory address space
- 64k × 16-bit external IO address space

- 32k × 16-bit external global memory address space
- 2k to 32k × 16-bit single-access On-chip PROM
- 1k to 9k × 16-bit single-access On-chip program/data RAM
- 1k × 16-bit dual-access On-chip program/data RAM
- Synchronous, TDM and buffered serial ports
- Programmable timer and PLL (Phase Locked Loops)
- IEEE standard JTAG ports
- 5 V/3 V operation with low power dissipation and power down modes
- DMA interface
- 100/128/132/144 pins in plastic QFP and TQFP

The various processors of TMS320C5x family and their characteristic features are listed in table 11.4.

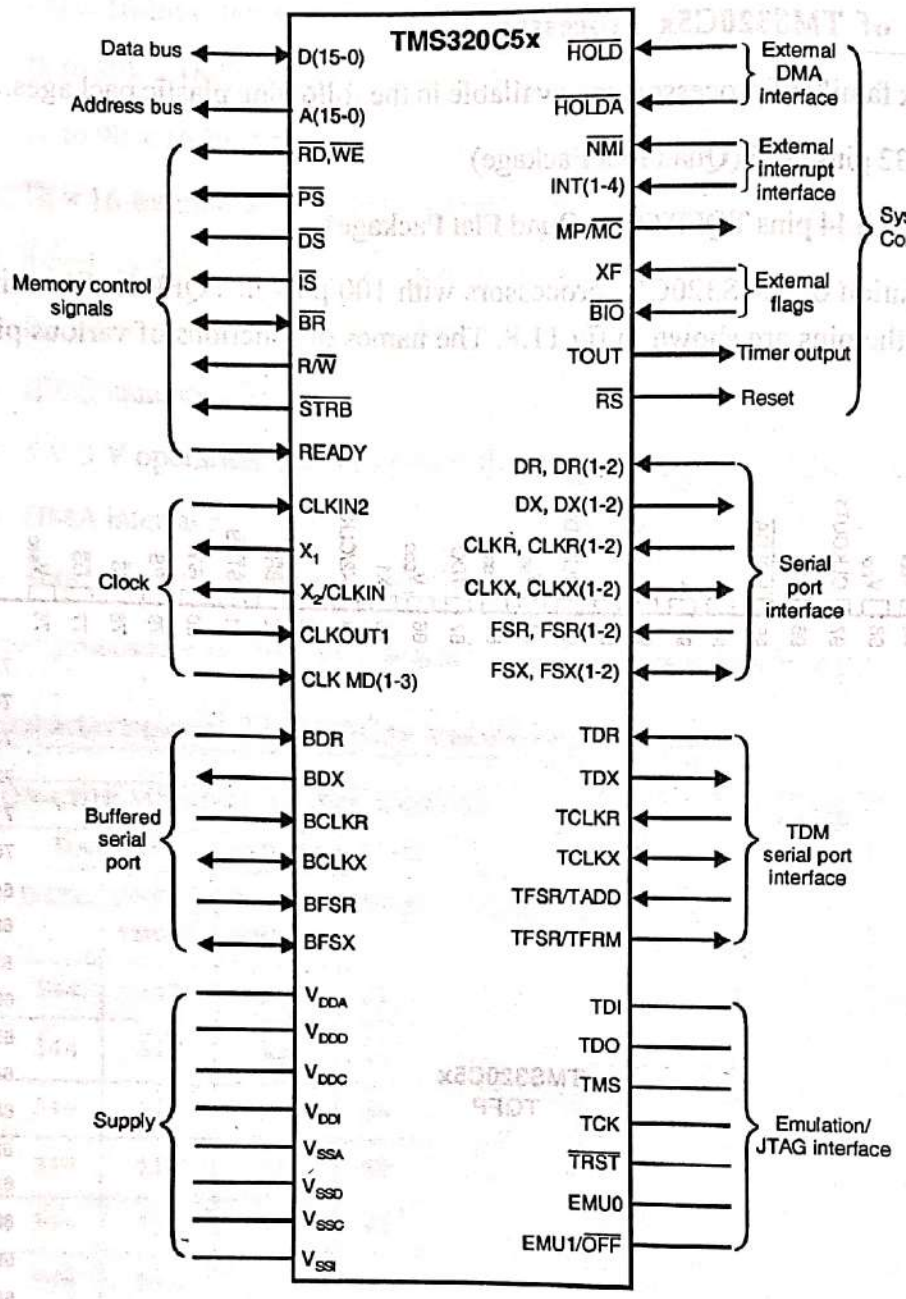


Fig 11.8 : Functional grouping of TMS320C5x pins (100 pin TQFP).

Table 11.5 : Pin Description of TMS320C5x (100 pin TQFP)

SIGNAL	TYPE	DESCRIPTION
	PARALLEL INTERFACE BUS	
A0 – A15	IO, Tristate	16-bit external address bus
D0 – D15	IO, Tristate	16-bit external data bus
$\overline{PS}$, $\overline{DS}$, $\overline{IS}$	Output, Tristate	Program, data and IO space select outputs, respectively
$\overline{STRB}$	IO, Tristate	Timing strobe for external cycles and external DMA
R / $\overline{W}$	IO, Tristate	Read/write select for external cycles and external DMA
$\overline{RD}$, $\overline{WE}$	Output, Tristate	Read and write strobes, respectively, for external cycles

Table 11.5 : Continued...

SIGNAL	TYPE	DESCRIPTION
READY	Input	External bus ready/wait-state control input
$\overline{BR}$	IO, Tristate	Bus request
SYSTEM INTERFACE/CONTROL SIGNALS		
$\overline{RS}$	Input	Reset
MP / $\overline{MC}$	Input	Microprocessor/microcomputer mode select
$\overline{HOLD}$	Input	Put parallel I/F bus in high-impedance state after current cycle
$\overline{HOLDA}$	Output, Tristate	Hold acknowledge
XF	Output, Tristate	External flag output
$\overline{BIO}$	Input	IO branch control input
TOUT	Output, Tristate	Timer output signal
$\overline{INT1}$ – $\overline{INT4}$	Input	External interrupt inputs
$\overline{NMI}$	Input	Nonmaskable external interrupt
SERIAL PORT INTERFACE		
DR, DR1, DR2	Input	Serial receive-data input
DX, DX1, DX2	Output, Tristate	Serial transmit-data output
CLKR, CLKR1, CLKR2	Input	Serial receive-data clock input
CLKX, CLKX1, CLKX2	IO, Tristate	Serial transmit-data clock
FSR, FSR1, FSR2	Input	Serial receive-frame-synchronization input
FSX, FSX1, FSX2	IO, Tristate	Serial transmit-frame-synchronization signal
BUFFERED SERIAL PORT (BSP)		
BDR	Input	BSP receive-data input
BDX	Output, Tristate	BSP transmit-data output
BCLKR	Input	BSP receive-data clock input
BCLKX	IO, Tristate	BSP transmit-data clock
BFSR	Input	BSP receive frame-synchronization input
BFSX	IO, Tristate	BSP transmit frame-synchronization signal
TDM SERIAL PORT INTERFACE		
TDR	Input	TDM serial receive-data input
TDX	Output, Tristate	TDM serial transmit-data output
TCLKR	Input	TDM serial receive-data clock input
TCLKX	IO, Tristate	TDM serial transmit-data clock
TFSR/TADD	IO, Tristate	TDM serial receive-frame-synchronization input
TFSX/TFRM	Input	TDM serial transmit-frame-synchronization signal

Table 11.5 : Continued...

SIGNAL	TYPE	DESCRIPTION
EMULATION/JTAG INTERFACE		
TDI	Input	JTAG-test-port scan data input
TDO	Output, Tristate	JTAG-test-port scan data output
TMS	Input	JTAG-test-port mode select input
TCK	Input	JTAG-port clock input
$\overline{\text{TRST}}$	Input	JTAG-port reset (with pull-down resistor)
EMU0	IO, Tristate	Emulation control 0
EMU1/$\overline{\text{OFF}}$	IO, Tristate	Emulation control 1
CLOCK GENERATION AND CONTROL		
X1	Output	Oscillator output
X2/CLKIN, CLKIN2	Input	Clock/oscillator input
CLKMD1, CLKMD2	Input	Clock-mode select inputs
CLKOUT1	Output, Tristate	Device system-clock output
POWER SUPPLY CONNECTIONS		
VDDA	Supply	Supply connection, address-bus output
VDDD	Supply	Supply connection, data-bus output
VDDC	Supply	Supply connection, control output
VDDI	Supply	Supply connection, internal logic
VSSA	Supply	Supply connection, address-bus output
VSSD	Supply	Supply connection, data-bus output
VSSC	Supply	Supply connection, control output
VSSI	Supply	Supply connection, internal logic

Table 11.6 : Device-Specific Pinouts for the 100 pin TQFP

PIN	TMS320C51 TMS320LC51	TMS320C52 TMS320LC52	TMS320C53S TMS320LC53S	TMS320LC56
5	TCLKX	V_{SSI}	CLKX2	BCLKX
6	CLKX	CLKX	CLKX1	CLKX
7	TFSR/TADD	V_{SSI}	FSR2	BFSR
8	TCLKR	V_{SSI}	CLKR2	BCLKR
46	DR	DR	DR1	DR
47	TDR	V_{SSI}	DR2	BDR
48	FSR	FSR	FSR1	FSR
49	CLKR	CLKR	CLKR1	CLKR

Table 11.6 : Continued...

PIN	TMS320C51 TMS320LC51	TMS320C52 TMS320LC52	TMS320C53S TMS320LC53S	TMS320LC56
83	CLKIN2	CLKIN2	CLKIN2	CLKMD3
91	FSX	FSX	FSX1	FSX
92	TFSX/TFRM	V_{SS}	FSX2	BFSX
93	DX	DX	DX1	DX
94	TDX	NC	DX2	BDX

11.3.2 Architecture of TMS320C5x Processors

The TMS320C5x processors have an advanced version of Harvard architecture, with separate buses for program and data, which facilitate simultaneous access of program and data. The program bus has separate lines to transmit data and address. Similarly the data bus has separate lines to transmit data and address.

The internal architecture of TMS320C5x processor is shown in fig 11.9. The architecture of TMS320C5x processors can be broadly divided into three major areas. They are, CPU (Central Processing Unit), memory and peripherals.

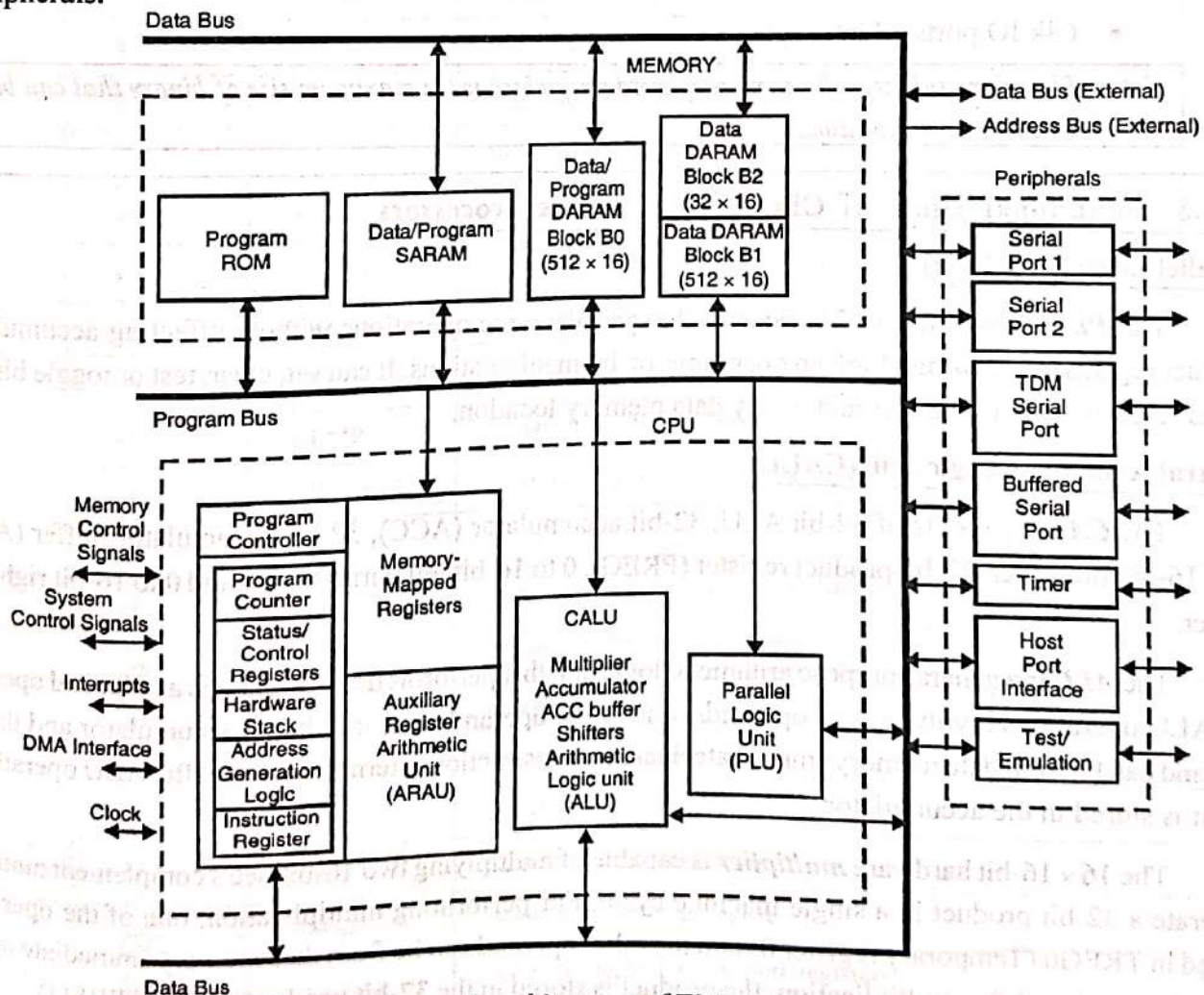


Fig 11.9 : Internal architecture of TMS320C5x.

The functional units of CPU are Parallel Logic Unit (PLU), central ALU, memory mapped registers, Auxiliary Register Arithmetic Unit (ARAU) and program controller.

The TMS320C5x processors has the following internal (or on-chip) memory.

- Program ROM (2k to 32k words)
- Data/Program Dual Access RAM (DARAM) (1024 + 32 = 1056 words)
- Data/Program Single Access RAM/(SARAM) (1k to 9k words)

The various on-chip or internal peripherals of TMS320C5x processors are clock generator, hardware timer, software programmable wait state generators, parallel IO ports, Host Port Interface (HPI), serial port, Buffered Serial Port (BSP), Time Division Multiplexed (TDM) serial port and user maskable interrupts.

The TMS320C5x processors have a total memory address space of 224k (including on-chip memory) with addressability of 16 bits. This address space is divided into four individually selectable address spaces as follows.

- 64k Program memory address space
- 64k Local data memory address space
- 32k Global data memory address space
- 64k IO ports address space

Note : The addressability refer to memory word size, which is the maximum size of binary that can be stored in one memory location.

11.3.3 Functional Units of CPU of TMS320C5x Processors

Parallel Logic Unit (PLU)

The ***PLU*** is an additional logic unit, that permits logic operations without affecting accumulator or product register. It performs Boolean operations or bit manipulations. It can set, clear, test or toggle bits in the status register, control register and in any data memory location.

Central Arithmetic Logic Unit (CALU)

The ***CALU*** consists of 32-bit ALU, 32-bit accumulator (ACC), 32-bit accumulator buffer (ACCB), 16 × 16-bit multiplier, 32-bit product register (PREG), 0 to 16-bit left barrel shifter and 0 to 16-bit right barrel shifter.

The ***ALU*** is a general-purpose arithmetic/logic unit that performs usual arithmetic and logical operations. For ALU operations involving two operands, one of the operands is stored in the accumulator and the other operand can be from data memory/immediate data in the instruction/internal register. After ALU operation, the result is stored in the accumulator.

The 16 × 16-bit hardware ***multiplier*** is capable of multiplying two 16-bit two's complement numbers to generate a 32-bit product in a single machine cycle. For performing multiplication, one of the operands is stored in TREG0 (Temporary register 0) and the other operand can be from data memory/immediate operand in the instruction. After multiplication, the product is stored in the 32-bit ***product register*** (PREG).

The 0 to 16-bit left and right shifter permit the content of memory to be shifted before loading into ALU and vice versa. The content of accumulator (ACC) and product register (PREG) can also be shifted using these shifters.

Memory-Mapped Registers

The TMS320C5x has 96 numbers of 16-bit memory-mapped registers, and they are mapped into page-0 of data memory space. The memory-mapped registers includes various control and status registers for CPU, serial port, timer and software wait-state generators. Also they include 16 memory-mapped IO ports. The memory-mapped registers along with their memory address are listed in table 11.7.

Table 11.7 : Memory-Mapped Registers of TMS320C5x Processors

ADDRESS		NAME	DESCRIPTION
DEC	HEX		
0–3	0–3	–	Reserved
4	4	IMR	Interrupt mask register
5	5	GREG	Global memory allocation register
6	6	IFR	Interrupt flag register
7	7	PMST	Processor mode status register
8	8	RPTC	Repeat counter register
9	9	BRCR	Block repeat counter register
10	A	PASR	Block repeat program address start register
11	B	PAER	Block repeat program address end register
12	C	TREG0	Temporary register 0 (used for multiplicand)
13	D	TREG1	Temporary register 1 (used for dynamic shift count)
14	E	TREG2	Temporary register 2 (used as bit pointer in dynamic bit test)
15	F	DBMR	Dynamic bit manipulation register
16–23	10–17	AR0–AR7	Auxiliary register 0 - Auxiliary register 7
24	18	INDX	Index register
25	19	ARCR	Auxiliary register compare register
26	1A	CBSR1	Circular buffer 1 start register
27	1B	CBER1	Circular buffer 1 end register
28	1C	CBSR2	Circular buffer 2 start register
29	1D	CBER2	Circular buffer 2 end register
30	1E	CBCR	Circular buffer control register
31	1F	BMAR	Block move address register
32–35	20–23	–	Memory-mapped serial port registers

Table 11.7 : Continued...

ADDRESS		NAME	DESCRIPTION
DEC	HEX		
36–42	24–2A	–	Memory-mapped peripheral registers
43–47	2B–2F	–	Reserved for test/emulation
48–55	30–37	–	Memory-mapped serial port registers
56–79	38–4F	–	Reserved
80–95	50–5F	–	Memory-mapped IO ports

Auxiliary Register Arithmetic Unit (ARAU)

The *ARAU* contains eight 16-bit auxiliary registers AR0-AR7, a 3-bit Auxiliary Register Pointer (ARP), a 16-bit index register (INDX) and a 16-bit Auxiliary Register Compare Register (ARCR). An unsigned 16-bit arithmetic unit in the ARAU is used to calculate indirect addresses, using the contents of ARP, INDX and ARCR registers. Therefore, the CALU is relieved from task of address manipulation and so it is free for other operations in parallel.

The auxiliary registers can also be used as general-purpose registers for holding the operands for arithmetic and logical operation in CALU.

Program Controller

The ***program controller*** contains logic circuits that decodes the instructions, manages the CPU pipeline, stores the status of CPU operations and decodes the conditional operations. Due to parallelism in architecture, the program controller can perform three concurrent or simultaneous memory operations in any given machine cycle. They are fetch an instruction, read an operand and write an operand.

The program controller unit consists of a 16-bit Program Counter (PC), 16-bit status registers ST0 and ST1, Processor Mode Status register (PMST) and Circular Buffer Control Register (CBCR), a 16 × 16-bit hardware stack, address generation logic, instruction register, interrupt flag register and interrupt mask register.

Status Registers (ST0 and ST1)

The TMS320C5x processors have two 16-bit ***status registers*** (ST0 and ST1) which holds the status of ALU result, pointers for indirect addressing and various bits for interrupt control, hold mode and product shift mode. The format of status registers are shown in fig 11.10 and 11.11. The functions of various bits of status registers are listed in Table 11.8 and 11.9.

The status registers can be stored into data memory and loaded from data memory, thereby allowing the processor status to be saved and restored for subroutines. The LST instruction writes to ST0 and ST1, and the SST instruction reads from them, except that the ARP bits and INTM bit are not affected by the LST #0 instruction.

The ST0 and ST1 each have an associated 1-level deep shadow register stack for automatic context-saving when an interrupt occurs. The INTM and OVM bits in ST0 and the C, CNF, HM, SXM, TC and XF bits in ST1 can be individually set using the SETC instruction or individually cleared using the CLRC instruction.

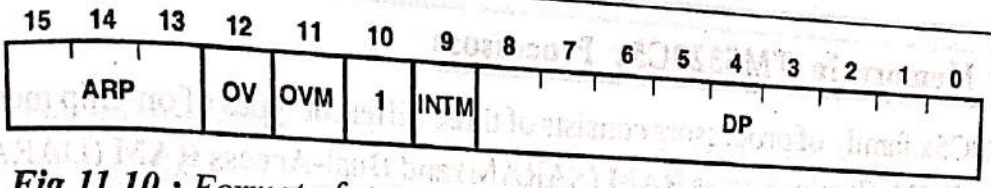


Fig 11.10 : Format of status register 0 (ST0) of TMS320C5x processors.

Table 11.8 : Functions of Various Bits of ST0 of TMS320C5x Processors

BIT	NAME	RESET VALUE	FUNCTION
15-13	ARP	X	Auxiliary register pointer to select AR for indirect addressing
12	OV	0	Overflow flag bit. Indicates an overflow in ALU operation
11	OVM	X	Overflow mode bit. Enables/disables the saturation mode in ALU
10	–	1	Always 1
9	INTM	1	Interrupt mode bit. Globally masks or enables all interrupts except NMI
8-0	DP	X	Data memory page pointer to specify the current data memory page

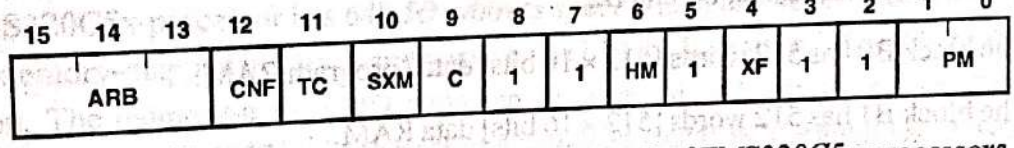


Fig 11.11 : Format of status register 1 (ST1) of TMS320C5x processors.

Table 11.9 : Function of Various Bits of ST1 of TMS320C5x Processors

BIT	NAME	RESET VALUE	FUNCTION
15-13	ARB	XXX	Auxiliary register buffer which holds the previous value in ARP of ST0
12	CNF	0	Configuration control bit to map Block 0 of DARAM either as data or program memory
11	TC	X	Test/control flag bit which stores the results of ALU or PLU test bit operations
10	SXM	1	Sign-extension mode bit. Enables/disables sign extension of an arithmetic operation
9	C	1	Carry bit which indicates a carry or borrow in ALU operation
8-7	–	11	Always 1
6	HM	1	Hold mode bit
5	–	1	Always 1
4	XF	1	Status of external flag pin
3-2	–	11	Always 1
1-0	PM	00	Product shift mode bits

11.3.4 On-Chip Memory in TMS320C5x Processors

The TMS320C5x family of processors consists of three different types of on-chip memory and they are mask-programmable ROM, Single-Access **RAM** (SARAM) and **Dual-Access RAM** (DARAM). The various members of TMS320C5x will have different capacity of on-chip memory which are listed in table 11.4.

Program ROM

The various models of TMS320C5x processors have internal maskable-Program **ROM** (PROM) of size 2k to 32k words. The processor has an option for including or excluding the on-chip PROM addresses in the processor program memory address space.

The main purpose of PROM is to permanently store the program code for a specific application during manufacturing of the chip itself. The processor has an option of boot loading the content of PROM to internal/external RAM during power-ON reset. The content of the PROM can be protected so that any external device cannot have access to the program code. This feature provides security for proprietary algorithms.

Data / Program Dual-Access RAM (DARAM)

The TMS320C5x processor has 1056 words [1056 × 16 bits] on-chip **Dual-Access** RAM (DARAM), which is divided into three blocks, B0, B1 and B2.

- The block B0 has 512 words [512 × 16 bits] data / program RAM.
- The block B1 has 512 words [512 × 16 bits] data RAM.
- The block B2 has 32 words [32 × 16 bits] data RAM.

Data / Program Single-Access RAM (SARAM)

The various models or TMS320C5x processor has 1k words to 9k words of SARAM.

The internal SARAM can be configured as data memory, program memory and combination of data and program memory

The SARAM can be divided into block of 1k/2k words with continuous address. The processor CPU can access one block for reading while writing in another block.

11.3.5 On-Chip Peripherals of TMS320C5x Processors

The various on-chip peripherals of TMS320C5x processors are clock generator, hardware timer, software-programmable wait-state generators, parallel IO ports, **Host Port Interface (HPI)** and serial ports.

Clock Generator

The ***clock generator*** of the TMS320C5x processor consists of an internal oscillator and a Phase Locked Loop (PLL) circuit. The clock generator can be driven by an external crystal resonator circuit or supplied by an external clock source. The ***PLL*** circuit can generate an internal CPU clock by multiplying the clock source by a specified factor, so that CPU is driven by high frequency clock and clock source can be used as source for other peripherals which runs at low frequency clock.

Hardware Timer

A 16-bit hardware timer with a 4-bit prescaler is available in TMS320C5x processor. This programmable *timer* generates clock at a rate that is between 1/2 and 1/32 of the machine cycle rate (CLKOUT1), depending upon the timer divide-down ratio. The timer can be stopped, restarted, reset or disabled by specific status bits. The processor has three registers to control and operate the timer and they are Timer Control Register (TCR), timer counter register (TIM) and timer period register (PRD). The timer counter register gives the current count of the timer. The timer period register defines the period for the timer. The 16-bit timer control register controls the operations of the timer.

Software Programmable Wait - State Generators

The TMS320C5x processor has software-programmable ***wait-state generators***, which can insert/ generate wait-states in external bus cycles for interfacing with slow speed external memory and IO devices. The processor consists of multiple wait-state generating circuits,and each circuit is user-programmable to insert different number of wait states for external memory accesses. These wait-state generators can extend the external bus cycles up to seven machine cycles.

Parallel IO Ports

The TMS320C5x processor has 64k ***IO address space*** which can be used as 64k IO ports and 16 of these ports are memory-mapped in data memory space. Each of the IO ports can be addressed by the IN or the OUT instruction. The memory-mapped IO ports can be accessed with any instruction that reads from or writes to data memory. TheTMS320C5x genearates a hardware signal $\overline{IS}$ during IO access to indicate a read or write operation through an IO port. The TMS320C5x can easily interface with external IO devices through the IO ports with minimal external address decoding circuits.

Host Port interface (HPI)

The HPI is available on the TMS320C57S and TMS320LC57 processors. The ***HPI*** is an 8-bit parallel IO port that provides an interface to a host processor for information exchange between the Digital Signal Processor (DSP) and the host processor. The DSP has 2k word on-chip memory that is accessible to both the host processor and the DSP. The HPI is connected to this memory through a dedicated bus, so that the CPU can work uninterrupted while the host processor accesses the memory through host port.

> ***Note : A host processor is an independent microprocessor/micorcontroller that is designed to carry out some specific tasks and deliver the results to digital signal processor.***

Serial Ports

Three different kinds of serial ports are available in TMS320C5x processors and they are general-purpose serial port, Time-Division Multiplexed (TDM) serial port and Buffered Serial Port (BSP). Every TMS320C5x processor contains at least one general-purpose, high-speed synchronous, full-duplexed serial port which can be used to provide direct communication with serial devices such as codec, serial analog-to digital (A/D) converters and other serial systems. The serial port is capable of operating at a clock rate up to one-fourth the machine cycle rate (CLKOUT1). The serial port transmitter and receiver are double-buffered and indiviually controlled by maskable external interrupt signals. For serial communication the data is framed either as bytes or as words.

The Buffered Serial Port (BSP) consists of a full-duplex double-buffered Serial Port Interface (SPI) and an Auto-Buffering Unit (ABU). The ABU allows the SPI to read/write directly to processor internal memory using a dedicated bus which enhances the speed of serial communication.

The Time-Division Multiplexed (TDM) serial ports can be used for serial communication between multiple processors. A maximum of eight processors having TDM ports can be connected via a pair of data lines and a pair of address lines for serial communication.

11.3.6 Addressing Modes of TMS320C5x Processors

The addressing mode is the method of specifying the data to be operated by an instruction. The TMS320C5x family of processors supports the following six addressing modes.

1. Direct addressing
2. Memory-mapped register addressing
3. Indirect addressing
4. Immediate addressing
5. Dedicated-register addressing
6. Circular addressing

Direct Addressing

In direct addressing, the lower 7 bits of data memory address are specified directly in the instruction itself. The upper 9 bits of the address will be the content of data memory page pointer (DP) in status register-0 (ST0).

Example :

```
ADDC 2Ch  ; Add the content of data memory whose page offset address
          ; (2Ch) is specified in the instruction and the carry to accumulator
          ; with sign extension suppressed.
```

Note : The TMS320C5x data memory is organized as 512 (or 2^9) pages with each page having 128 (or 2^7) locations ($2^9 \times 2^7 = 2^{16} = 64k$). The DP holds the 9-bit current page address and the 7-bit address in the instruction is the page offset address.

Memory-Mapped Register Addressing

In memory-mapped register addressing, the address of the memory-mapped register can be specified as direct address in the instruction. The memory-mapped register addressing is a special case of direct addressing in which only page offset address is used to access the memory and the default page address is 000h. Therefore, the data pointer need not be loaded with page address for this addressing mode.

Example :

```
LAMM 16h  ; Load accumulator with the content of memory-mapped
          ; register mapped to address 0016h.
SAMM 16h  ; Store the content of accumulator in memory-mapped
          ; register mapped to address 0016h.
```

Note : The memory-mapped registers of TMS320C5x are mapped to page-0 of data memory address space.

Indirect Addressing

In indirect addressing mode, the data memory address is specified by the content of one of the eight auxiliary registers, AR0 - AR7. The AR (Auxiliary Register) currently used for accessing data is denoted by ARP (Auxiliary Register Pointer).

In indirect addressing mode, the content of AR can be updated automatically either after or before the operand is fetched. The syntax used in the operand field of instruction for modifying the content AR are listed in table 11.10.

Example :

```
LACC *,0      ; Load the content of data memory addressed by AR, (which in turn pointed by ARP) to
              ; accumulator without left shift. AR is not altered.
LACC *+, 0    ; Same as above, but AR is incremented by one.
LACC *–, 0    ; Same as above, but AR is decremented by one.
LACC *0+, 0   ; Same as above, but AR is incremented by the value in index register.
LACC *0–, 0   ; Same as above, but AR is decremented by the value in index register.
```

Table 11.10 : Syntax Used in Indirect Address for Modifying AR

Syntax	Modification of AR
*	AR unaltered
*+	AR incremented by 1
*–	AR decremented by 1
*0+	AR incremented by the content of index register
*0–	AR decremented by the content of index register
*BR0+	AR incremented for bit reversed addressing using the content of index register
*BR0–	AR decremented for bit reversed addressing using the content of index register

Bit Reversed Addressing

In bit reversed addressing, the data memory address is specified by AR like indirect addressing, but the content of AR is incremented/decremented in order to generate the data memory address in the bit reversed order, using the content of index register. (The bit reversed addressing is a special case of indirect addressing).

Example :

```
MAC 14F0h, *BR0+  ; The content of program memory is multiplied by the data memory and the product is
                  ; added to the accumulator. The address of the program memory is 14F0h (which is
                  ; specified in the instruction). The address of data memory is the content of AR currently
                  ; pointed by ARP. The AR is incremented to generate the bit reversed address of data
                  ; memory operand.
```

Immediate Addressing

In immediate addressing, the data is specified as a part of the instruction. In this addressing, the instruction will carry an 8-bit/9-bit/13-bit/16-bit constant, which is the data to be operated by the instruction. The immediate constant is specified with # symbol.

Example :

```
ADD # 4Ah     ; Add the immediate, 8-bit data 4Ah given in the instruction to accumulator.
ADD # 8C4Ah   ; Add the immediate 16-bit data 8C4Ah given in the instruction to accumulator.
```

Dedicated Register Addressing

In dedicated register addressing mode, the address of one of the operands is specified by a dedicated CPU register BMAR (**Block Move Address Register**). In this addressing mode, the address of the memory block to be accessed can be changed during execution of the program.

In another case of dedicated register addressing, one of the operands is the content of a dedicated CPU register DBMR (**Dynamic Bit Manupulation Register**).

Example :

```
BLDD BMAR, 6Fh  ; The source address is the content of BMAR. The lower 7 bits of destination address is
                ; 6Fh, (which is directly given in the instruction) and the upper 9 bits of destination
                ; address is the content of DP. This instruction will copy the content of source address to
                ; destination address.

OPL  6Fh        ; The content of DBMR is logically ORed with the content of data memory specified by the
                ; instruction and result is stored back in data memory. The lower 7 bits of data memory
                ; is 6Fh (which is directly given in the instruction) and the upper 9 bits is the
                ; content of DP.
```

Circular Addressing

The circular addressing is similar to indirect addressing. This addressing mode allows the specified memory buffer to be accessed sequentially with a pointer that automatically wraps around to the beginning of the buffer when the last location is accessed.

In circular addressing mode, when the address pointer is incremented, the address in AR will be checked with the end address of the circular buffer, and if it exceeds the end address then the address is made equal to start address of the circular buffer.

In order to hold the start and end addresses of the circular buffer, the TMS320C5x has four circular buffer registers, namely,

CBSR1 : **Circular Buffer-1 Start address Register**

CBSR2 : **Circular Buffer-2 Start address Register**

CBER1 : **Circular Buffer-1 End address Register**

CBER2 : **Circular Buffer-2 End address Register**

With the help of the above registers, at any one time, two circular buffers can be defined. A Circular Buffer Control Register (CBCR) is used to enable/disable the circular buffers.

11.3.7 Instruction Pipelining in TMS320C5x Processors

The execution of TMS320C5x processor instructions involve four levels/phases of pipelining. The four phases of pipelining are fetch, decode, read and execute. The functions performed in the four phases are given below.

Fetch : The opcode (instruction code) is fetched from program memory and PC is updated.

Decode : The opcode is decoded, the data address is generated and data address generation registers are updated.

Read : The operand is read from data memory.

Execute : Perform the task specified by the instruction.

The TMS320C5x employs a four-phase clock so that one phase of four consecutive instructions can be processed simultaneously. When the first instruction is in execute phase, the second instruction will be in read phase, the third instruction will be in decode phase and the fourth instruction will be in fetch phase. Therefore, the average execution time of one word instruction will be one clock cycle. While executing two word instructions and branch instructions, some of the phases will be idle for some time and so average execution time will be more than one clock cycle.

11.3.8 Instructions of TMS320C5x Processors

The TMS320C5x processors instruction set consists of instructions that supports both numeric-intensive signal processing operations and general-purpose applications. The instructions can be classified into following groups.

1. Arithmetic instructions
2. Logical instructions
3. Branch/control instructions
4. Load/store instructions
5. Block move instructions

The instructions of TMS320C5x processors are classified into the above groups, arranged in alphabetical order and listed in table 11.11.

The size of TMS320C5x instructions is either 1 word or 2 words. When all the instructions and data reside in internal memory, most of the instructions are executed in one or two clock cycles. The execution time for some of the data transfer, branch and MAC instructions is 3 to 4 clock cycles.

Table 11.11 : Instruction Set Summary for TMS320C5x

ARITHMETIC INSTRUCTIONS	
INSTRUCTION	**DESCRIPTION**
ABS	Absolute value of ACC
ADCB	Add ACCB to ACC with carry
ADD *dma [,shift]*	Add memory to ACC with shift (Direct addressing)
ADD *{ind} [,shift [,next ARP]]*	Add memory to ACC with shift (Indirect addressing)
ADD # *k*	Add short immediate to low ACC
ADD # *lk [,shift2]*	Add long immediate to ACC with shift
ADDB	Add ACCB to ACC